Answer on Question #57100-Math-Calculus

Prove that ∫∫∫ (x2+y2+z2)dxdydz=a5/20 at x=0 to a-x-y, y=0 to a-x and z=0 t0 a.

Solution

$$\int_0^a dz \int_0^{a-z} dy \int_0^{a-z-y} dx\,(x^2+y^2+z^2) = \int_0^a dz \int_0^{a-z} dy \left[\frac{x^3}{3} + x(y^2+z^2)\right]_0^{a-z-y}$$

$$= \int_0^a dz \int_0^{a-z} dy \frac{(a-z-y)}{3}\,[(a-z-y)^2 + 3(y^2+z^2)]$$

$$= \int_0^a dz \int_0^{a-z} dy \left[\left(\frac{a^3}{3} - a^2 z + 2az^2 - \frac{4z^3}{3}\right) + y(2az - a^2 - 2z^2) + y^2(2a-2z) - \frac{4}{3}y^3\right]$$

$$= \int_0^a dz \left[\left(\frac{a^3}{3} - a^2 z + 2az^2 - \frac{4z^3}{3}\right) y + \frac{y^2}{2}(2az - a^2 - 2z^2) + \frac{y^3}{3}(2a-2z) - \frac{1}{3}y^4\right]_0^{a-z}$$

$$= \int_0^a dz \left[\left(\frac{a^3}{3} - a^2 z + 2az^2 - \frac{4z^3}{3}\right)(a-z) + \frac{(a-z)^2}{2}(2az - a^2 - 2z^2)\right.$$

$$\left. + \frac{(a-z)^3}{3}(2a-2z) - \frac{1}{3}(a-z)^4\right] = \int_0^a dz \left[\frac{a^4}{6} - \frac{2}{3}a^3 z + \frac{3}{2}a^2 z^2 - \frac{5}{3}az^3 + \frac{2}{3}z^4\right]$$

$$= \left[\frac{a^4}{6}z - \frac{1}{3}a^3 z^2 + \frac{1}{2}a^2 z^3 - \frac{5}{3}a\frac{z^4}{4} + \frac{2}{3}\frac{z^5}{5}\right]_0^a = a^5\left(\frac{1}{6} - \frac{1}{3} + \frac{1}{2} - \frac{5}{3\cdot 4} + \frac{2}{3\cdot 5}\right) = \frac{a^5}{20}.$$