Answer on Question #57099 – Math – Calculus

Question

Q. prove that ∫∫∫ (x2+y2+z2)dzdydx=a5/20 at x=0 to a-x-y, y=0 to a-x and z=0 t0 a.

Solution

There should be z=0 to a-x-y, y=0 to a-x and x=0 to a.

Let us evaluate the given integral $\int\int\int(x^2+y^2+z^2)\,dzdydx$.

$$\int\int\int(x^2+y^2+z^2)\,dzdydx$$

$$=\int_0^a dx\left(\int_0^{a-x} dy\left(\int_0^{a-x-y}(x^2+y^2+z^2)dz\right)\right)$$

$$=\int_0^a dx\left(\int_0^{a-x} dy\left(\left[x^2z+y^2z+\frac{z^3}{3}\right]\Big|_0^{a-x-y}\right)\right)$$

$$=\int_0^a dx\left(\int_0^{a-x}\left(x^2(a-x-y)+y^2(a-x-y)+\frac{1}{3}(a-x-y)^3\right)dy\right)$$

$=\int_0^a dx\left(\int_0^{a-x}\left(x^2(a-x)+\frac{1}{3}(a-x)^3-x^2y+(a-x)y^2-y^3+\frac{1}{3}(-3(a-x)^2y+3(a-x)y^2-y^3)\right)dy\right)=\int_0^a\left[x^2(a-x)y+\frac{1}{3}(a-x)^3y-\frac{x^2y^2}{2}+(a-x)\frac{y^3}{3}-\frac{y^4}{4}-(a-x)^2\frac{y^2}{2}+(a-x)\frac{y^3}{3}-\frac{y^4}{12}\right]\Big|_0^{a-x}dx=\int_0^a\left[x^2(a-x)(a-x)+\frac{1}{3}(a-x)^3(a-x)-\frac{x^2(a-x)^2}{2}+(a-x)\frac{(a-x)^3}{3}-\frac{(a-x)^4}{4}-(a-x)^2\frac{(a-x)^2}{2}+(a-x)\frac{(a-x)^3}{3}-\frac{(a-x)^4}{12}-0\right]dx=\int_0^a\left[x^2(a-x)^2+\frac{(a-x)^4}{3}-\frac{x^2(a-x)^2}{2}+\frac{(a-x)^4}{3}-\frac{(a-x)^4}{4}-\frac{(a-x)^4}{2}+\frac{(a-x)^4}{3}-\frac{(a-x)^4}{12}\right]dx=\int_0^a\left[\left(1-\frac{1}{2}\right)x^2(a-x)^2+\left(\frac{1}{3}+\frac{1}{3}-\frac{1}{4}-\frac{1}{2}+\frac{1}{3}-\frac{1}{12}\right)(a-x)^4\right]dx=\frac{1}{2}\int_0^a x^2(a-x)^2dx+\frac{1}{6}\int_0^a(a-x)^4dx=\frac{1}{2}\int_0^a x^2(a^2-2ax+x^2)dx-\frac{1}{6}\frac{(a-x)^5}{5}\Big|_0^a=\frac{1}{2}\left(a^2\frac{x^3}{3}-2a\frac{x^4}{4}+\frac{x^5}{5}\right)\Big|_0^a+\frac{a^5}{30}=\left(\frac{1}{6}-\frac{1}{4}+\frac{1}{10}+\frac{1}{30}\right)a^5=\frac{10-15+6+2}{60}a^5=$

$=\frac{a^5}{20}.$