Answer on Question #57081 – Math – Calculus

Question

If

$$I_n = \int_0^{\pi/2} x^n \sin x \; dx\,, \quad n \geq 1$$

show that $I_{n+1} - (n+1)\left(\frac{\pi}{2}\right)^n = -n(n+1)I_{n-1}$. Also deduce the value of I_4.

Solution

Calculate I_{n+1} using the integration by parts.

Let $u = x^{n+1},\;\; dv = \sin x\, dx,$ then $du = (n+1)x^n dx,\;\; v = -\cos x.$

$$I_{n+1} = \int_0^{\pi/2} x^{n+1} \sin x \; dx = -x^{n+1}\cos x\Big|_0^{\pi/2} - \left(-(n+1)\int_0^{\pi/2} x^{n-1}\cos x \; dx\right)$$

$$= -\left(\frac{\pi}{2}\right)^{n+1} \cdot \cos\frac{\pi}{2} + 0 \cdot 1 + (n+1)\int_0^{\pi/2} x^{n-1}\cos x \; dx = (n+1)\int_0^{\pi/2} x^n \cos x \; dx$$

because $\cos\frac{\pi}{2} = 0.$

Integrate by parts once more with $\mathrm{u} = \mathrm{x}^{\mathrm{n}},\;\; \mathrm{dv} = \cos \mathrm{x}\, \mathrm{dx}$ and $\mathrm{du} = \mathrm{n}\mathrm{x}^{\mathrm{n}-1}\mathrm{dx},\;\; \mathrm{v} = \sin \mathrm{x}$:

$$I_{n+1} = (n+1)\int_0^{\pi/2} x^n \cos x \; dx = (n+1)\left(x^n \sin x\Big|_0^{\pi/2} - n\int_0^{\pi/2} x^{n-1}\sin x \; dx\right) =$$

$$= (n+1)\left[\left(\frac{\pi}{2}\right)^n \cdot 1 - 0\right] - (n+1)n\int_0^{\pi/2} x^{n-1}\sin x \; dx = (n+1)\left(\frac{\pi}{2}\right)^n - (n+1)n \cdot I_{n-1}, \qquad (1)$$

because the last integral equals $I_{n-1} = \int_0^{\pi/2} x^{n-1}\sin x \; dx.$

From this equality we obtain

$$I_{n+1} - (n+1)\left(\frac{\pi}{2}\right)^n = -n(n+1)I_{n-1},$$

hence

$$I_{n+1} = -n(n+1)I_{n-1} + (n+1)\left(\frac{\pi}{2}\right)^n, \qquad (2)$$

where

$I_0 = \int_0^{\pi/2} \sin x \; dx = -\cos x\Big|_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = -0 + 1 = 1,$

$$I_1 = \int_0^{\frac{\pi}{2}} x\sin x\,dx = |u = x, dv = \sin x\,dx,\ \ du = dx,\ \ v = -\cos x| = -x\cos x|_0^{\frac{\pi}{2}} +$$

$$+\int_0^{\frac{\pi}{2}} \cos x\,dx = \sin x|_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$$

So using the formula (2)

$I_2 = -1 \cdot 2 \cdot I_0 + 2\frac{\pi}{2} = -2 + \pi = \pi - 2$ and

$$I_4 = -3 \cdot 4 \cdot I_2 + 4\left(\frac{\pi}{2}\right)^3 = 4\left(\frac{\pi}{2}\right)^3 - 4 \cdot 3 \cdot (\pi - 2) = \frac{\pi^3}{2} - 12\pi + 24.$$

Answer: $I_4 = \frac{\pi^3}{2} - 12\pi + 24.$