Answer on Question#57023 – Math – Calculus

Question. Use Parseval's theorem to find the value of the sum $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$

Solution. Consider the next function defined on $[-\pi;\ \pi]$: $f(x)=\begin{cases}1, & 0\le|x|\le\frac{\pi}{2},\\ 0, & \frac{\pi}{2}\le|x|\le\pi\end{cases}$. Since f is even then $b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$ and $a_0=\frac{4}{2\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\cdot\frac{\pi}{2}=1$;

$a_n=\frac{4}{2\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi/2}\cos nx\,dx=\frac{2}{\pi n}\sin nx\,\Big|_{x=0}^{\pi/2}=\frac{2}{\pi n}\sin\frac{\pi n}{2}, n\in\mathbb{N}.$

So the Fourier's decomposition of the function f has the next form:

$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{\pi n}{2}}{n}\cos nx$. From the Parseval's theorem it follows that

$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx$. In our case we have:

$\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\sin\frac{\pi n}{2}\right)^2}{n^2}=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}dx=1\Rightarrow\sum_{n=1}^{\infty}\frac{\left(\sin\frac{\pi n}{2}\right)^2}{n^2}=\frac{\pi^2}{8}.$

Since $\left(\sin\frac{\pi n}{2}\right)^2=\begin{cases}0, & n=2k\\ 1, & n=2k+1\end{cases}$, $k=0,1,2,\ldots$ then we obtain that

$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$

Answer. $\frac{\pi^2}{8}$.