Answer on Question #57008 – Math – Calculus

Question

State and prove Divergence theorem.

Solution

The Divergence Theorem

If $\underline{a}$ is a vector field in a volume *V*, and *S* is a closed surface bounding *V*, then

$$\int_V \underline{\nabla} \cdot \underline{a} dV = \int_S \underline{a} \cdot \underline{dS}$$

Proof. We derive the divergence theorem by making use of the integral definition of $\underline{\nabla} \cdot \underline{a}$

$$\underline{\nabla} \cdot \underline{a} = \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \underline{a} \cdot \underline{dS}$$

Since this **definition** of $\underline{\nabla} \cdot \underline{a}$ is valid for volumes of arbitrary shape, we can build a smooth surface *S* from a large number *N* of blocks of volume $\Delta V^{(i)}$ and surface $\Delta S^{(i)}$. We have

$$\underline{\nabla} \cdot \underline{a}\left(\underline{r^{(i)}}\right) = \frac{1}{\Delta V^{i}} \int_{\Delta S^{i}} \underline{a} \cdot \underline{dS} + O\left(e^{i}\right)$$

where $e^{(i)} \to 0$ as $\Delta V^{(i)} \to 0$. Now multiply both sides by $\Delta V^{(i)}$ and sum over all *i*

$$\sum_{i=1}^{N} \underline{\nabla} \cdot \underline{a}\left(\underline{r^{(i)}}\right) \Delta V^{(i)} = \sum_{i=1}^{N} \int_{\Delta S^{(i)}} \underline{a} \cdot \underline{dS} + \sum_{i=1}^{N} e^{(i)} \Delta V^{(i)}$$

On rhs the contributions from surface elements interior to S cancel. This is because where two blocks touch, the outward normals are in opposite directions, implying that the contributions to the respective integrals cancel. Taking the limit $N \to \infty$ we have, as claimed,

$$\int_V \underline{\nabla} \cdot \underline{a} dV = \int_S \underline{a} \cdot \underline{dS}.$$