Answer on Question #56979 – Math – Analytic Geometry

Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2,5).

Solution

Method 1

The equation of the tangent is given by

$$y = y(x_0) + y'(x_0)(x - x_0)$$

If $y^2 = 12x$, then $y = \sqrt{12x}$ or $y = -\sqrt{12x}$.

Case 1. Assume $y = \sqrt{12x}$. Then

$$y' = \left(\sqrt{12x}\right)' = \sqrt{12}\left(x^{1/2}\right)' = \sqrt{12} \cdot \frac{1}{2} \cdot x^{-1/2} = \sqrt{\frac{3}{x}}$$

$$y(x_0) = \sqrt{12x_0}$$

$y'(x_0) = \sqrt{\frac{3}{x_0}}, x_0 \neq 0.$

If $x_0 = 0$, then the derivative y' is not finite at point $x_0 = 0$, so the equation of tangent is $x = 0$, but this tangent does not passes through the point (2,5), therefore we regect the case, where $x_0 = 0$. Thus, $x_0 \neq 0$.

So

$y = \sqrt{12x_0} + \sqrt{\frac{3}{x_0}}(x - x_0).$

Besides,

$$5 = \sqrt{12x_0} + \sqrt{\frac{3}{x_0}}(2 - x_0),$$

because the tangent line passes through the point (2,5),

Multiplying both sides by $x_0 \neq 0$,

$$5 \cdot \sqrt{x_0} = 2\sqrt{3} \cdot x_0 + \sqrt{3}(2 - x_0)$$

$$\sqrt{3}x_0 - 5\sqrt{x_0} + 2\sqrt{3} = 0$$

Substitute $\sqrt{x_0} = t \geq 0$, then

$$\sqrt{3}t^2 - 5t + 2\sqrt{3} = 0$$

$$D = 25 - 4 \cdot \sqrt{3} \cdot 2\sqrt{3} = 25 - 24 = 1$$

$$t_1 = \frac{5 - 1}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$t_1 = \frac{5 + 1}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Hence $x_0 = (t_1)^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$ or $x_0 = (t_2)^2 = \left(\sqrt{3}\right)^2 = 3$

Thus,

$$y=\sqrt{12\cdot\frac{4}{3}}+\sqrt{\frac{3\cdot 3}{4}}\left(x-\frac{4}{3}\right)$$

or

$$y=\sqrt{12\cdot 3}+\sqrt{\frac{3}{3}}(x-3).$$

They are equivalent to

$$y=2+\frac{3}{2}x$$

or

$$y=x+3.$$

Case 2. Assume $y=-\sqrt{12x}$. Then

$$y'=-\left(\sqrt{12x}\right)'=-\sqrt{12}\left(x^{\frac{1}{2}}\right)'=-\sqrt{12}\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}=-\sqrt{\frac{3}{x}}$$

$$y(x_0)=-\sqrt{12x_0}$$

$$y'(x_0)=-\sqrt{\frac{3}{x_0}}$$

So

$$y=-\sqrt{12x_0}-\sqrt{\frac{3}{x_0}}(x-x_0)$$

$5=-\sqrt{12x_0}-\sqrt{\frac{3}{x_0}}(2-x_0),$

because the tangent line passes through the point (2,5).

Multiplying both sides by $x_0\neq 0,$

$$5\cdot\sqrt{x_0}=-2\sqrt{3}\cdot x_0-\sqrt{3}(2-x_0)$$

$$-\sqrt{3}x_0-5\sqrt{x_0}-2\sqrt{3}=0$$

Substituting $\sqrt{x_0}=t\geq 0$, get

$\sqrt{3}t^2+5t+2\sqrt{3}=0,$

$D=25-4\cdot\sqrt{3}\cdot 2\sqrt{3}=25-24=1,$

$t_1=\frac{-5-1}{2\sqrt{3}}=\frac{-3}{\sqrt{3}}=-\sqrt{3}<0,$

$t_1=\frac{-5+1}{2\sqrt{3}}=\frac{-2}{\sqrt{3}}<0,$

Hence this case does not satisfy assumption $\sqrt{x_0}=t\geq 0.$

Method 2

Equation of tangent line of $y^2 = 2px$ at point (x_0, y_0) is $yy_0 = p(x + x_0)$, where $y_0^2 = 2px_0$.

It follows $x_0 = \frac{y_0^2}{12}$ from $y_0^2 = 12x_0$.

In this problem $y^2 = 12x,\ p = 6$ and $(x, y) = (2,5)$ satisfies the equation $yy_0 = p(x + x_0)$, that is,

$$5y_0 = 6(2 + x_0)$$

Substituting for $x_0 = \frac{y_0^2}{12}$ into $5y_0 = 6(2 + x_0)$ get $5y_0 = 6\left(2 + \frac{y_0^2}{12}\right)$, hence

$$5y_0 = 6\frac{1}{12}(24 + y_0^2),$$

$$y_0^2 - 10y_0 + 24 = 0,$$

$$(y_0 - 6)(y_0 - 4) = 0,$$

therefore $y_0 = 6$ or $y_0 = 4$, hence

$$x_0 = \frac{y_0^2}{12} = \frac{6^2}{12} = \frac{36}{12} = 3$$

or

$$x_0 = \frac{y_0^2}{12} = \frac{4^2}{12} = \frac{16}{12} = \frac{4}{3}$$

Finally there exists two tangents:

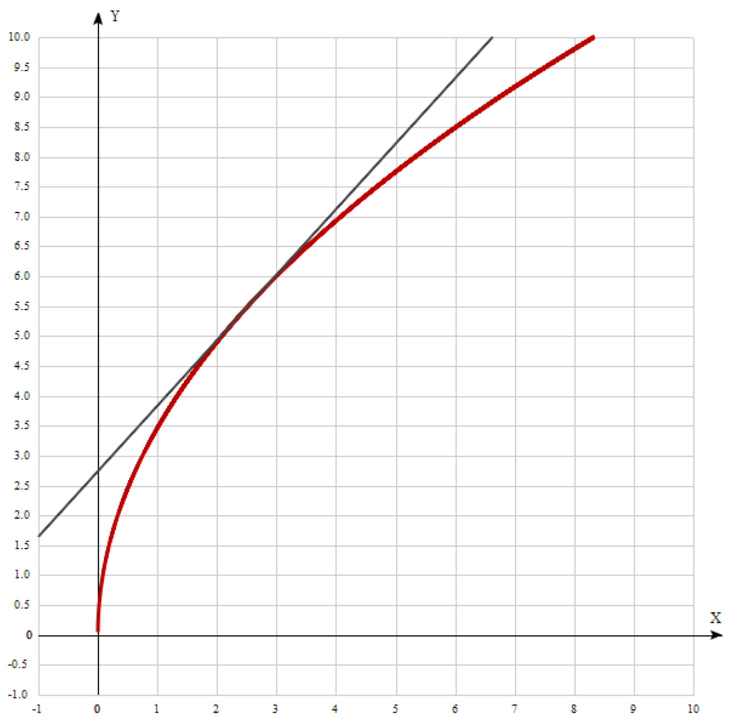
substituting for $(x_0, y_0) = (3,6)$ into $yy_0 = p(x + x_0)$ get $6y = 6(x + 3)$, that is, $y = x + 3$;

substituting for $(x_0, y_0) = \left(\frac{4}{3}, 4\right)$ into $yy_0 = p(x + x_0)$ get $4y = 6\left(x + \frac{4}{3}\right)$, that is, $y = \frac{3}{2}x + 2$.

Remark

The equations of the tangent of the parabola $y^2 = 12x$ at the point, where x=2.5, are

$$Y = \sqrt{30} + \sqrt{\frac{3}{2.5}}(x - 2.5)$$



and

$$Y = -\sqrt{30} - \sqrt{\frac{3}{2.5}}(x - 2.5)$$