Answer on Question #56927 – Math – Linear Algebra

Question

Check signs definiteness
$y = f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 3x_2^2$

Solution

First method

The matrix of quadratic form $y = f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 3x_2^2$ is

$$M = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$$

In mathematics, Sylvester's criterion is a necessary and sufficient criterion to determine whether a Hermitian matrix is positive definite.

Sylvester criterion states that a Hermitian matrix *M* is positive definite if and only if all the following matrices have a positive determinant:

the upper left 1-b-1 corner of *M*

the upper left 2-by-2 corner of *M*

…

M itself.

In other words, all of the leading principal minors must be positive.

Calculate $\Delta_{\mathbf{1}} = 2 > 0$, $\Delta_{\mathbf{2}} = \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} = 2 \cdot 3 - (-2) \cdot (-2) = 6 - 4 = 2 > 0.$

Because all of the leading principal minors are positive, a matrix and the quadratic form are positive definite according to Sylvester's criterion.

Second method

If *M* is the symmetric matrix that defines the quadratic form, and *S* is any invertible matrix such that $D = SMS^T$ is diagonal, then the number of negative elements in the diagonal of *D* is always the same, for all such *S*, and the same goes for the number of positive elements according to Sylvester's law of inertia.

It holds true that the symmetric matrix *M* is positive definite if and only if all its eigenvalues are strictly positive.

To transform the given quadratic form into a diagonal form, find eigenvalues of the matrix

$$M = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$$

by solving the equation

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

We have $(2-\lambda)(3-\lambda)-4=\lambda^2-5\lambda+2=0$

This quadratic equation has two roots

$\lambda_1=\frac{5-\sqrt{17}}{2}$ and $\lambda_2=\frac{5+\sqrt{17}}{2}$.

Since $\sqrt{17}$ is greater than 4 and less than 5, both roots are positive. So coefficients in diagonal form are strictly positive. Hence the given quadratic form is also positive definite.

Answer: the form $f(x_1,x_2)=2x_1^2-4x_1x_2+3x_2^2$ is positive definite.

Third method

$y=f(x_1,x_2)=2x_1^2-4x_1x_2+3x_2^2=2(x_1^2-2x_1x_2+x_2^2)+x_2^2=2(x_1-x_1)^2+x_2^2>0$ for $(x_1;x_2)\neq(0;0)$, hence the form is positive definite by definition.