Problem.

Consider the family of lines, 5x + 3y − 2 + K1 (3x − y − 4) = 0 and x − y + 1 + K2(2x − y − 2)=0. Find the equation of the line belonging to both the families without determining their vertices.

Solution

Reduce the given families of lines to the standard form $y = ax + b$:

$$\begin{cases} 5x+3y-2+k_1(3x-y-4)=0 \\ x-y+1+k_2(2x-y-2)=0 \end{cases}$$

$$\begin{cases} y(3-k_1)=(-5-3k_1)x+2+4k_1 \\ y(1+k_2)=(1+2k_2)x+1-2k_2 \end{cases}$$

$$\begin{cases} y=-\dfrac{5+3k_1}{3-k_1}x+\dfrac{2+4k_1}{3-k_1} \\ y=\dfrac{1+2k_2}{1+k_2}x+\dfrac{1-2k_2}{1+k_2} \end{cases}$$

Both of these equations will describe the same line, if

$$\begin{cases} \dfrac{5+3k_1}{k_1-3}=\dfrac{1+2k_2}{k_2+1} \\ \dfrac{2+4k_1}{3-k_1}=\dfrac{1-2k_2}{k_2+1} \end{cases}$$

We have

$$k_1 \neq 3, \quad k_2 \neq -1, \qquad (*)$$

$$\begin{cases} (5+3k_1)(k_2+1)=(1+2k_2)(k_1-3) \\ (2+4k_1)(k_2+1)=(1-2k_2)(3-k_1) \end{cases}$$

$$\begin{cases} 5+5k_2+3k_1+3k_1k_2=k_1+2k_1k_2-3-6k_2 \\ 2k_2+4k_1k_2+2+4k_1=3-6k_2-k_1+2k_1k_2 \end{cases}$$

$$\begin{cases} k_1k_2+11k_2+2k_1=-8 \\ 2k_1k_2+8k_2+5k_1=-17 \end{cases}$$

Multiply the first equation by −2 and add with the second one

$$-14k_2+k_1=17$$

$$k_1=14k_2+17$$

Substituting this value in the first equation

$$(14k_2+17)k_2+11k_2+2(14k_2+17)=-8$$

$$14k_2^2+17k_2+11k_2+28k_2=-8$$

$$14k_2^2+56k_2+42=0$$

$$k_2^2 + 4k_2 + 3 = 0$$

$$(k_2)_1 = -1$$

$$(k_2)_2 = -3$$

The value $k_2 = -1,$ doesn't satisfy (*).

If $k_2 = -3$ then

$$k_1 = 14(-3) + 17 = -25$$

$$a = \frac{1 + 2k_2}{1 + k_2} = \frac{1 + 2(-3)}{(-3) + 1} = \frac{5}{2}$$

$$b = \frac{1 - 2(-3)}{1 + (-3)} = -\frac{7}{2}$$

Hence, the needed line is

$$y = \frac{5}{2}x - \frac{7}{2}$$

$$\underline{5x - 2y - 7 = 0}$$