Answer the question #56866 – Math – Analytic Geometry

If lines be drawn parallel to the axes of coordinates from the points where $x\cos\alpha + y\sin\alpha = p$ meets them so as to meet the perpendicular on this line from the origin in the points *P* and *Q* then prove that $|PQ| = 4p|\cos 2\alpha| \cdot \operatorname{cosec}^2(2\alpha)$.

Solution

Let AB is the line, which shows by the given equation $x\cos\alpha + y\sin\alpha = p$ (see Fig. below). Then coordinates of the points A and B are $\left(0; \frac{p}{\sin\alpha}\right)$ and $\left(\frac{p}{\cos\alpha}; 0\right)$ accordingly. After that we write the equation for the line PQ which is perpendicular to AB and crosses the origin $(0;\ 0)$. We obtain

$PQ: x\sin\alpha - y\cos\alpha = 0.$

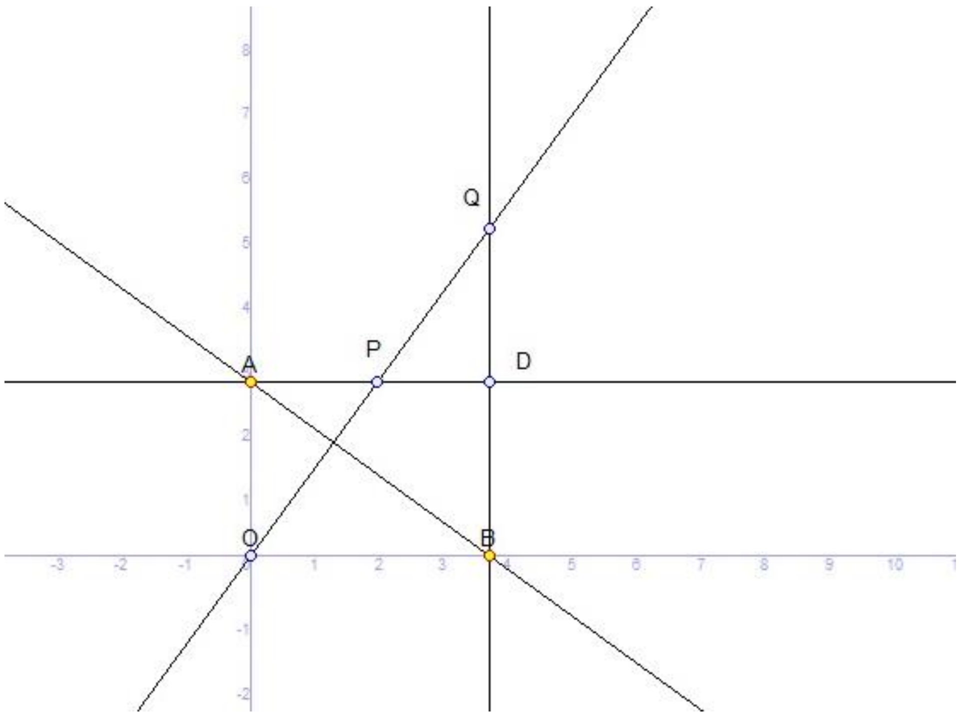
Equation of line *AD* is $y = \frac{p}{\sin\alpha}$, hence the y-coordinate of point *P* is $y_P = \frac{p}{\sin\alpha}$.

Equation of line *BD* is $x = \frac{p}{cos\alpha}$, hence the x-coordinate of point *Q* is $x_p = \frac{p}{cos\alpha}$. Coordinates of the points P and Q satisfy equation $PQ: x\sin\alpha - y\cos\alpha = 0$, so they are $\left(\frac{p\cos\alpha}{\sin^2\alpha}; \frac{p}{\sin\alpha}\right)$ and $\left(\frac{p}{\cos\alpha}; \frac{p\sin\alpha}{\cos^2\alpha}\right)$ respectively. By the formula $|PQ| = \sqrt{\left(x_q - x_p\right)^2 + \left(y_q - y_p\right)^2}$, we obtain:

$$|PQ| = \sqrt{\left(\frac{p}{\cos\alpha} - \frac{p\cos\alpha}{\sin^2\alpha}\right)^2 + \left(\frac{p\sin\alpha}{\cos^2\alpha} - \frac{p}{\sin\alpha}\right)^2}$$

$$= p\sqrt{\left(\frac{\sin^2\alpha - \cos^2\alpha}{\cos\alpha\sin^2\alpha}\right)^2 + \left(\frac{\sin^2\alpha - \cos^2\alpha}{\cos^2\alpha\sin\alpha}\right)^2} = p\sqrt{\frac{\cos^2(2\alpha)}{\cos^2\alpha\sin^4\alpha} + \frac{\cos^2(2\alpha)}{\cos^4\alpha\sin^2\alpha}}$$

$$= \frac{4p|\cos(2\alpha)|}{\sqrt{16\cos^4\alpha\sin^4\alpha}} = \frac{4p|\cos(2\alpha)|}{\sqrt{(\sin(2\alpha))^4}} = 4p|\cos(2\alpha)| \cdot \operatorname{cosec}^2(2\alpha)$$



Thus, $|PQ| = 4p|\cos 2\alpha| \cdot \operatorname{cosec}^2(2\alpha)$.
