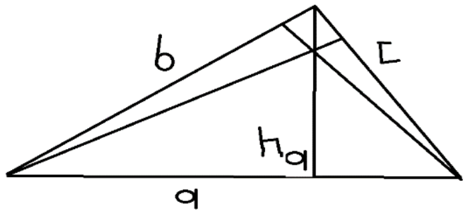
Answer on Question #56865 – Math – Analytic Geometry

Find the co-ordinates of the orthocentre of the triangle, the equations of whose sides are x + y = 1,2x + 3y = 6, 4x − y + 4 = 0, without finding the co−ordinates of its vertices.

Solution



Rewrite equations x + y = 1,2x + 3y = 6, 4x − y + 4 = 0 as $y = -x + 1$, $y = -\frac{2}{3}x + 2$, $y = 4x + 4$ respectively.

Given:

$$a: y = -x + 1$$
$$b: y = -\frac{2}{3}x + 2$$
$$c: y = 4x + 4$$

Three lines $y = m_1x + b_1, y = m_2x + b_2, y = m_3x + b_3$ intersect at a single point $(x^*; y^*)$.

Given m_1, b_1, m_2, b_2, m_3. Find b_3.

To find the point of intersection of straight lines

$$y = m_1x + b_1, y = m_2x + b_2, \qquad (1)$$

equate the right-hand sides of the previous equalities:

$m_1x + b_1 = m_2x + b_2,$

$(m_1 - m_2)x = b_2 - b_1$

$\Rightarrow x = -\frac{b_1 - b_2}{m_1 - m_2}.$

Substitute for x into $y = m_1x + b_1$:

$y = m_1x + b_1,$

$y = m_1 \cdot \left(-\frac{b_1 - b_2}{m_1 - m_2}\right) + b_1,$

$y = \frac{-m_1b_1 + m_1b_2 + b_1m_1 - b_1m_2}{m_1 - m_2},$

$y = \frac{m_1b_2 - b_1m_2}{m_1 - m_2},$

$y = \frac{b_1m_2 - b_2m_1}{m_2 - m_1}.$

Thus, the point of intersection of straight lines (1) is

$$(x^*;\ y^*) = \left(-\frac{b_1 - b_2}{m_1 - m_2}; \frac{b_1m_2 - b_2m_1}{m_2 - m_1}\right). \qquad (2)$$

The line $y = m_3x + b_3$ also goes through point (2), therefore $y^* = m_3x^* + b_3$, hence

$$b_3 = y^* - m_3x^* = \frac{b_1m_2 - b_2m_1}{m_2 - m_1} - m_3 \cdot \left(-\frac{b_1 - b_2}{m_1 - m_2}\right)$$

$$b_3 = \frac{b_1(m_2 - m_3) - b_2(m_1 - m_3)}{m_2 - m_1}. \quad (3)$$

When two straight lines are perpendicular, the product of their slopes is (-1).

If equation of side *a* is $y = -x + 1$, then its slope is $k_1 = -1$, hence the slope of perpendicular straight line is $k_2 = \frac{-1}{k_1} = \frac{-1}{-1} = 1$ and equation of the height drawn to the side *a* is

$$h_a:\ y = x + b_a \quad (4)$$

If equation of side *b* is $y = -\frac{2}{3}x + 2$, then its slope is $k_3 = -\frac{2}{3}$, hence the slope of perpendicular straight line is $k_4 = \frac{-1}{k_3} = \frac{-1}{-2/3} = \frac{3}{2}$ and equation of the height drawn to the side *b* is

$$h_b: y = \frac{3}{2}x + b_b \quad (5)$$

Straight lines h_a, b, c intersect at a single point. Using formula (3), put $m_1 = -\frac{2}{3}, b_1 = 2,\ m_2 = 4, b_2 = 4, m_3 = 1$ and we can find b_a:

$$b_a = \frac{2(4-1) - 4(-\frac{2}{3} - 1)}{4 + \frac{2}{3}} = \frac{19}{7}. \quad (6)$$

Straight lines h_b, a, c intersect at a single point. Using formula (3) put $m_1 = -1, b_1 = 1,\ m_2 = 4, b_2 = 4, m_3 = \frac{3}{2}$ and we can find b_b:

$$b_b = \frac{1(4 - \frac{3}{2}) - 4(-1 - \frac{3}{2})}{4 + 1} = 2.5. \quad (7)$$

From (4), (5), (6), (7) it follows that

$$h_a: y = x + \frac{19}{7}, \quad (8)$$

$$h_b: y = \frac{3}{2}x + 2.5 \quad (9)$$

Orthocentre:is the point of intersection of heights h_a, h_b. Using (8) and (9) obtain the following system of equations:

$$\begin{cases} y = x + \frac{19}{7} \\ y = \frac{3}{2}x + 2.5 \end{cases} \to \frac{3}{2}x + 2.5 = x + \frac{19}{7} \to \left(\frac{3}{2} - 1\right)x = \frac{19}{7} - 2.5 \to$$

$$\to \mathrm{x} = \frac{3}{7}\ ;\ \mathrm{y} = x + \frac{19}{7} = \frac{3}{7} + \frac{19}{7} = 3\frac{1}{7}$$

Thus,$\left(\frac{3}{7}; 3\frac{1}{7}\right)$ is orthocenter.

Answer: $\left(\frac{3}{7}, 3\frac{1}{7}\right)$.