Answer on Question #56726 – Math – Differential Equations

Question

$\frac{dy}{dx}=-\frac{4+xy}{x^2}+y^2$. Find a solution to the ordinary differential equation in the form

$$y=\frac{a}{x}$$

Solution

$y'=-\frac{4}{x^2}-\frac{y}{x}+y^2$. Let $y_1=\frac{a}{x}$ be a solution of equation. Then

$-\frac{a}{x^2}=-\frac{4}{x^2}-\frac{a}{x^2}+\frac{a^2}{x^2}$,

$a^2=4$,

hence $a=\pm 2$.

Case 1

Let $a=2$.

Substituting $y=z+\frac{2}{x}$ we obtain

$z'-\frac{2}{x^2}=z^2+\frac{4z}{x}+\frac{4}{x^2}-\frac{z}{x}-\frac{2}{x^2}-\frac{4}{x^2}$;

$z'-\frac{3z}{x}=z^2$;

$\frac{z'}{z^2}-\frac{3}{zx}=1$.

Substituting $t=\frac{1}{z}$, $t'=-\frac{1}{z^2}z'$.

$t'+\frac{3t}{x}=-1$

First we solve $\hat{t}'+\frac{3\hat{t}}{x}=0$;

$\frac{d\hat{t}}{\hat{t}}=-\frac{3dx}{x}$; $\ln\hat{t}=-3\ln x+\ln C$; $\hat{t}=\frac{C}{x^3}$.

Let $t=\frac{\varphi(x)}{x^3}$, then $\frac{\varphi'(x)x^3-\varphi(x)3x^2}{x^6}+\frac{3\varphi(x)}{x^4}=-1$.

Hence $j'(x) = -x^3$, which gives $j(x) = -\frac{x^4}{4} + C_1$, where C_1 is an arbitrary real constant.

Thus, $t = -\frac{x}{4} + \frac{C_1}{x^3} = \frac{-x^4 + 4C_1}{4x^3} = \frac{x^4 - 4C_1}{-4x^3} = \frac{x^4 + C}{-4x^3}$.

Get back to substitution $t = \frac{1}{z}$ and obtain $z = \frac{-4x^3}{x^4 + C}$.

Since $y = z + \frac{2}{x}$, then $y = \frac{-4x^3}{x^4 + C} + \frac{2}{x} = \frac{2C - 2x^4}{x(x^4 + C)}$, where C is an arbitrary real constant.

<u>Case 2</u>

Let $a = -2$.

Substituting $y = z - \frac{2}{x}$ we obtain

$$z' + \frac{2}{x^2} = z^2 - \frac{4z}{x} + \frac{4}{x^2} - \frac{z}{x} + \frac{2}{x^2} - \frac{4}{x^2}$$

$$z' = z^2 - \frac{5z}{x}$$

$$\frac{z'}{z^2} + \frac{5}{zx} = 1.$$

Substituting $t = \frac{1}{z}$, $t' = -\frac{1}{z^2} z'$.

$$t' + \frac{5t}{x} = 1$$

First we solve

$$\hat{t}' + \frac{5\hat{t}}{x} = 0$$

$$\frac{d\hat{t}}{t} = -\frac{5dx}{x},$$

$$\ln \hat{t} = -5\ln x + \ln C,$$

$$\hat{t} = \frac{C}{x^5}.$$

Let $t = \frac{j(x)}{x^5}$, then $\frac{j'(x)x^5 - j(x)5x^4}{x^{10}} + \frac{5j(x)}{x^6} = 1.$

Hence $j'(x) = x^5$, which gives $j(x) = \frac{x^6}{6} + C_1$, where C_1 is an arbitrary real constant, thus, $t = \frac{x}{6} + \frac{C_1}{x^5} = \frac{x^6 + 6C_1}{6x^5} = \frac{x^6 + C}{6x^5}$.

Get back to substitution $t = \frac{1}{z}$ and obtain

$$z = \frac{6x^5}{x^6 + C}.$$

Since $y = z - \frac{2}{x}$, then $y = \frac{6x^5}{x^6 + C} - \frac{2}{x} = \frac{6x^6 - 2x^6 - 2C}{x(x^6 + C)} = \frac{4x^6 - 2C}{x(x^6 + C)}$, where C is an arbitrary real constant.

Answer: $y_1 = \frac{2C_1 - 2x^4}{x(x^4 + C_1)}$, $y_2 = \frac{4x^6 - 2C_2}{x(x^6 + C_2)}$, where C_1, C_2 are arbitrary real constants.