Answer on Question #56702 – Math – Linear Algebra

Question

Find a basis for the subspace W of R4, spanned by the set of vectors V1 {[1 1 0 -1]}, V2 {[0 1 2 1]}, V3 {[1 0 1 -1]}, V4 {[1 1-6 -3]} and V5 {[-1 -5 1 0]}.

Solution

Given the set

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -6 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

of vectors in the vector space R^4, find a basis for span S.

The set $W = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ of vectors in R^4 is **linearly independent** if the only solution of

(*) $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = 0$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ -6 \\ -3 \end{bmatrix} + c_5 \begin{bmatrix} -1 \\ -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1= c_2= c_3= c_4= c_5=0.$

Rearranging the left hand side yields

$$\begin{bmatrix} 1\, c_1 + 0\, c_2 + 1\, c_3 + 1\, c_4 - 1\, c_5 \\ 1\, c_1 + 1\, c_2 + 0\, c_3 + 1\, c_4 - 5\, c_5 \\ 0\, c_1 + 2\, c_2 + 1\, c_3 - 6\, c_4 + 1\, c_5 \\ -1\, c_1 + 1\, c_2 - 1\, c_3 - 3\, c_4 + 0\, c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix equation above is equivalent to the following **homogeneous system of equations**

(**)

$$
\begin{aligned}
1\,c_1 + 0\,c_2 + 1\,c_3 + 1\,c_4 - 1\,c_5 &= 0\\
1\,c_1 + 1\,c_2 + 0\,c_3 + 1\,c_4 - 5\,c_5 &= 0\\
0\,c_1 + 2\,c_2 + 1\,c_3 - 6\,c_4 + 1\,c_5 &= 0\\
-1\,c_1 + 1\,c_2 - 1\,c_3 - 3\,c_4 + 0\,c_5 &= 0
\end{aligned}
$$

1	0	1	1	-1
1	1	0	1	-5
0	2	1	-6	1
-1	1	-1	-3	0

can be transformed by a sequence of elementary row operations to the matrix

1	0	0	3	-4
0	1	0	-2	-1
0	0	1	-2	3
0	0	0	0	0

The reduced row echelon form of the coefficient matrix of the homogeneous system (**) is

1	0	0	3	-4
0	1	0	-2	-1
0	0	1	-2	3
0	0	0	0	0

which corresponds to the system

$$
\begin{aligned}
1\,c_1 \qquad\qquad + 3\,c_4 - 4\,c_5 &= 0\\
1\,c_2 \qquad - 2\,c_4 - 1\,c_5 &= 0\\
1\,c_3 - 2\,c_4 + 3\,c_5 &= 0\\
0 &= 0
\end{aligned}
$$

The leading entries have been highlighted in yellow.

Those columns in the matrix, that do not contain leading entries, correspond to unknowns that will be arbitrary. The system has **infinitely many solutions**:

$c_1 = -3\ c_4 \quad +4\ c_5$
$c_2 = 2\ c_4 \quad +1\ c_5$
$c_3 = 2\ c_4 \quad -3\ c_5$
$c_4 =$ *arbitrary*
$c_5 =$ *arbitrary*

Since the variables c_4, c_5 are arbitrary, then each of the vectors $\mathbf{v}_4$, $\mathbf{v}_5$ can be expressed as a linear combination of vectors in the set T = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$}.

For example, set $c_4 = 1$, $c_5 = 0$, then

$c_1 =-3$, $c_2 =2$, $c_3 =2$.

Use the equation (*) to express $\mathbf{v}_4$ as a linear combination of the remaining vectors in the set S.

Let

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = 0$,

$(-3\ c_4+4\ c_5)\ \mathbf{v}_1+ (2\ c_4+1\ c_5)\mathbf{v}_2 + (2\ c_4-3\ c_5)\mathbf{v}_3 + 1\mathbf{v}_4 + 0\mathbf{v}_5 =0$.

Put $c_4 = 1$, $c_5 = 0$, then

$-3\ \mathbf{v}_1+ 2\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4 + 0\mathbf{v}_5 =0$,

hence

$\mathbf{v}_4 = 3\mathbf{v}_1-2\mathbf{v}_2-2\mathbf{v}_3$.

If $c_4 = 0$, $c_5 = 1$, then

$c_1 = 4$, $c_2 =1$, $c_3 =-3$.

Let

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = 0$,

$(-3\ c_4+4\ c_5)\ \mathbf{v}_1+ (2\ c_4+1\ c_5)\mathbf{v}_2 + (2\ c_4-3\ c_5)\mathbf{v}_3 + 1\mathbf{v}_4 + 0\mathbf{v}_5 =0$.

Put $c_4 = 0$, $c_5 = 1$, then

$4\mathbf{v}_1 + \mathbf{v}_2 + -3\mathbf{v}_3 + 0\mathbf{v}_4 + \mathbf{v}_5 = 0$,

hence

$\mathbf{v}_5 = -4\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$.

Since the set T = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is **linearly independent** and it **spans span W**, then the set

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

forms a basis for span W.

Answer:

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Question

What is the dimension of W?

Solution

The dimension of W is 3 (number of nonzero rows).

1	0	0	3	-4
0	1	0	-2	-1
0	0	1	-2	3
0	0	0	0	0

Answer: 3.

Question

Let W be the subspace of P3 spanned by:
$\{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, 2t^3 + 3t^2 - 4t + 3\}$.
(i) Find a basis for W. **(ii)** What is the dimension of W?

Solution

1	0	1	2
1	1	0	3
-2	0	-2	-4
1	1	0	3

1	0	1	2
0	1	-1	1
0	0	0	0
0	1	-1	1

1	0	1	2
0	1	-1	1
0	0	0	0
0	0	0	0

As the columns of matrix containing leading entries are the first and the second columns, the first and the second polynomial form a basis for W. That is, the set

$\{t^3 + t^2 - 2t + 1, t^2 + 1\}$ is a basis for W. Thus the dimension of W is 2.

Answer: (i) $\{t^3 + t^2 - 2t + 1, t^2 + 1\}$; **(ii)** 2.