Answer on Question #56698 – Math – Combinatorics |Number Theory

13. Let three lines L_1, L_2, L_3 be given by $2x + 3y = 2, 4x + 6y = 5, 6x + 9y = 10$ respectively. If line L_r contains 2^r different points on it, where $r \in \{1,2,3\}$, then maximum number of triangles which can be formed with vertices at the given points on the lines, are given by:

(a) 320 (b)304 (c) 364 (d) 360

Solution

Lines L_r, $r \in \{1,2,3\}$ are parallel. L_1 contains 2 points, L_2 contains 4 points and L_3 contains 8 points.

To form triangles we must take 3 points and these 3 points can't lie on one line.

We use combinations, because the order of points isn't important for us:

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Possible cases:

1. 2 points from L_1, 1 point from L_2.
   The number of such triangles is equal to $C_2^2 C_4^1 = 1 \cdot \frac{4!}{3!} = 4$.
2. 2 points from L_1, 1 point from L_3.
   The number of such triangles is equal to $C_2^2 C_8^1$ =1·8=8.
3. 1 point from L_1, 2 points from L_2.
   The number of such triangles is equal to $C_2^1 C_4^2$ =2·6=12.
4. 1 point from L_1, 2 points from L_3.
   The number of triangles: $C_2^1 C_8^2$ =2·28 =56.
5. 1 point from L_1, 1 point from L_2, 1 point from L_3.
   The number of such triangles is equal to $C_2^1 C_4^1 C_8^1$ =2·4·8=64.
6. 2 points from L_2, 1 point from L_3.
   The number of such triangles is equal to $C_4^2 C_8^1$ =6·8=48.
7. 1 point from L_2, 2 points from L_3.
   The number of such triangles is equal to $C_4^1 C_8^2$ =4·28=112.

The total number of triangles is equal to

N=4+8+12+56+64+48+112=304

N=$C_2^2(C_4^1 + C_8^1) + C_2^1(C_4^2 + C_8^2 + C_4^1 C_8^1) + C_4^2 C_8^1 + C_4^1 C_8^2 =$

= 12+132+48+112= 304.

Answer: (b) 304.

20. Total number of ways of selecting two numbers from the set of {1,2,3,4,…,3n} so that their sum is divisible by 3 is equal to:

(a) $3n^2 - n$ (b) $\frac{3n^2-n}{2}$ (c) $\frac{2n^2-n}{2}$ (d) $2n^2 - n$

Solution

Let S_n be the total number of ways of selecting two numbers from the 3n elements.

S_n= C_{3n}^2 = $\frac{3n(3n-1)}{2!}$, n=2,3, … Then S_2=15, S_3=36, etc.

Let A_n is a set of such pairs of number which sum is divisible by 3, n=2,3,…

Let $a = 3m + k$ and $b = 3n + l$, then

$$a + b = (3m + k) + (3n + l) = (3m + 3n) + k + l$$

Is divisible by 3 if $k + l$ is divisible by 3.

Then

A_2= ({1,2}, {1,5}, {2,4}, {4,5}, {3,6}), the number of its elements is $|A_2|$=5;

A_3= A_2 ∪ ({1,8}, {2,7}, {3,9}, {4,8}, {5,7}, {7,8}, {6,9}), the number of its elements is $|A_3|$=12

and so on.

We have the following answer:

$|A_n|$=$\frac{S_n}{3}$= $\frac{3n^2-n}{2}$.

Answer: (b) $\frac{3n^2-n}{2}$.