Answer on Question #56687 – Math – Combinatorics | Number Theory

6. If the L.C.M. of $'\alpha'$ and $'\beta'$ is $p^2q^4r^3$, where p, q, r are prime numbers and $\alpha, \beta \in I^+$, then the number of ordered pairs $(\alpha, \ \beta)$ are:

(a) 225 (b) 420 (c) 315 (d) 192

Solution

Let us consider the numbers α and β in the form $p^{a'}q^{b'}r^{c'}$, where p, q, r are prime numbers and a', b', c' are nonnegative integers. Let their LCM be equal to $p^aq^br^c$ that is a', b', c' must belong to the sets $0 \le a' \le a, 0 \le b' \le b, 0 \le c' \le c$. Let in the pair (α, β) with $\text{LCM}(\alpha, \beta) = p^aq^br^c$ the multipliers $A = q^{b'}r^{c'}, B = q^{b''}r^{c''}$ be fixed (here $\alpha = p^{a'}A$ and $\beta = p^{a''}B$). Then numbers with such a form can have $\text{LCM}(\alpha, \beta) = p^aq^br^c$ in the following cases: either

$$a' = a, \quad 0 \le a'' \le a, \quad \text{i.e. } a + 1 \text{ numbers,}$$

or

$$0 \le a' \le a, \quad a'' = a, \qquad \text{i.e. } a + 1 \text{ numbers,}$$

(the pair with a'=a, a''=a enters into both sets).

Consequently, the number of pairs (α, β) with fixed A and B is equal to

$$2(a + 1) - 1 = 2a + 1,$$

where we have excluded the pair with a'=a, a''=a that has been counted twice.

In a similar way the number of pairs $\alpha = q^{b'}C, \beta = q^{b''}D$ (C, D fixed) is equal to $2b + 1$ and the number of pairs $\alpha = r^{c'}E, \beta = r^{c''}F$ (E, F fixed) is equal to $2c + 1$.

Since p, q, r are prime numbers, the above choices

$$a' = a, \quad 0 \le a'' \le a$$

$$0 \le a' \le a, \quad a'' = a$$

and similar choices for b', b'', c', c'', can be made independently and therefore the number of ordered pairs (α, β) with $\text{LCM}(\alpha, \beta) = p^aq^br^c$ is equal to

$$P(a, b, c) = (2a + 1)(2b + 1)(2c + 1).$$

In the suggested question $a = 2, b = 4, c = 3$ and we get

$$P = 5 \cdot 9 \cdot 7 = 315.$$

Answer: (c), number of ordered pairs is equal to 315.

7. Total number of non-negative integral solutions of $18 < x_1 + x_2 + x_3 \leq 20$, is given by:

(a) 1245 (b) 685 (c) 1150 (d) 441

Solution

This number is equal to the sum of numbers of solutions of two equations

$$x_1 + x_2 + x_3 = 19,$$

or

$$x_1 + x_2 + x_3 = 20.$$

Let us consider the first one: $x_1 + x_2 + x_3 = 19$.

If $x_1 = 0$, then $x_2 + x_3 = 19$ has 20 solutions (0,19), (1,18), …, (18,1),(19,0).

If $x_1 = 1$, then $x_2 + x_3 = 18$ has 19 solutions (0,18), (1,17), …, (17,1),(18,0).

…

If $x_1 = 19$, then $x_2 + x_3 = 0$ has 1 solution (0,0).

Thus, the number of solutions of this equation is equal to

$$20 + 19 + \cdots + 1 = \frac{20 \cdot 21}{2} = 210.$$

In a similar way, the number of solutions of the equation $x_1 + x_2 + x_3 = 20$ is equal to

$$21 + 20 + \cdots + 1 = \frac{21 \cdot 22}{2} = 231.$$

Consequently, the total number of solutions of inequality $18 < x_1 + x_2 + x_3 \leq 20$ is equal to $210 + 231 = 441$.

Answer: (d), total number of non-negative integral solutions is equal to 441.