QUESTION

If x and y belong to real numbers, show that

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

SOLUTION

Firstly it must be said that the

$$\forall x, y \in \Re \qquad |x| \ge 0,\ |y| \ge 0 \implies 1+|x| \ge 1,\ \ 1+|y| \ge 1,\ 1+|x+y| \ge 1$$

The proof will be constructed as follows: make a chain of identical transformations to get the performance of a simple inequality which data $x, y \in \Re$ obvious. As the $|x|+1 \ge 1$ counter is positive, multiplying both sides of the inequality in the $|x|+1$ will not change the sign of inequality

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}\Bigg| (1+|x|)*(1+|y|)*(1+|x+y|) \iff$$

$$\frac{|x+y|*(1+|x|)*(1+|y|)*(1+|x+y|)}{1+|x+y|} \le$$

$$\le \frac{|x|*(1+|x|)*(1+|y|)*(1+|x+y|)}{1+|x|} + \frac{|y|*(1+|x|)*(1+|y|)*(1+|x+y|)}{1+|y|};$$

$$(1+|x|)*(1+|y|)*|x+y| \le |x|*(1+|y|)*(1+|x+y|) + |y|*(1+|x|)*(1+|x+y|);$$

$$(1+|x|+|y|+|x|*|y|)*|x+y| \le |x|*(1+|y|+|x+y|+|y|*|x+y|)+|y|*(1+|x|+|x+y|+|x|*|x+y|);$$

$$|x+y|+|x|*|x+y|+|y|*|x+y|+|x|*|y|*|x+y| \le |x|+|x|*|y|+|x|*|x+y|+|x|*|y|*|x+y|+$$

$$+|y|+|y|*|x|+|y|*|x+y|+|y|*|x|*|x+y|;$$

$$|x+y| \le |x|+|y|+|x|*|y|*|x+y|+2*|x|*|y|$$

Using the triangle inequality

$$|x+y| \le |x|+|y|,\ \ \forall x, y \in \Re$$

$$|x+y| \le |x|+|y|+|x|*|y|*|x+y|+2*|x|*|y| \quad \iff \quad 0 \le |x|*|y|*|x+y|+2*|x|*|y|$$

As the last inequality $0 \le |x|*|y|*|x+y|+2*|x|*|y|\ \forall x, y \in \Re$ is satisfied, then the initial and inequality

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} \quad \forall x, y \in \Re$$