Answer on Question #56669 – Math – Real Analysis

Between any two distinct real numbers there exist an infinite many irrational numbers. Prove this statement.

Solution

First, let's prove that there exists at least one rational number between two distinct real numbers.

Let $x, y \in \mathbb{R}$ such that $x < y$. Let's bound them by natural number $M \in \mathbb{N}$: $|x|, |y| < M$. Then let

$$x' = \frac{x+M}{2M} \in (0,1)$$
$$y' = \frac{y+M}{2M} \in (0,1)$$

Consider also the average of two: $z = \frac{x+y}{2}$ and

$$z' = \frac{z+M}{2M} \in (0,1)$$

Obviously, $x < z < y$.

Consider their decimal representations:

$$x' = 0.\alpha_1\alpha_2\alpha_3 \ldots$$
$$z' = 0.\gamma_1\gamma_2\gamma_3 \ldots$$
$$y' = 0.\beta_1\beta_2\beta_3 \ldots$$

Since x and z are distinct, so are x' and z'. Therefore there exists such index $k \in \mathbb{N}$ that digits from index 1 to $k-1$ are equal: $\alpha_i = \gamma_i, i < k$, and digit at k-th position differ: $\alpha_k < \gamma_k$. Then the rational number $t = 0.\gamma_1 \ldots \gamma_k$ (truncation of z) is greater than x, but not greater than z, therefore

$$x' < t \leq z' < y'$$

Reverse the transformation to $(0,1)$ interval:

$$x < 2Mt - M < y$$

$r = 2Mt - M$ is rational since t is rational and M is natural. Thus, between any two real numbers x, y there exists a rational number.

Consider the following process. At first iteration for two distinct real numbers $x < y$ we obtain a rational number between them: $x < r_1 < y$. At second iteration we obtain a rational number between x and r_1: $x < r_2 < r_1 < y$. At third – a rational number between x and r_2:

$x < r_3 < r_2 < r_2 < y$. And so on. Due to this process we have an infinite number of rational numbers $r_1, r_2, \ldots$ lying between x and y. Thus, there exists infinitely many rational numbers between any two distinct real numbers.