Answer on Question #56647 – Math – Combinatorics | Number Theory

3.

Statement-I. The greatest value of ${}^{40}C_0\,{}^{60}C_r + {}^{40}C_1\,{}^{60}C_{r-1} + \dots + {}^{40}C_{40}\,{}^{60}C_{r-40}$ is ${}^{100}C_{50}$

Because

Statement-II. The greatest value of ${}^{2n}C_r$ (where *r* is constant) occurs at $r = n$

Solution. Indeed, ${}^{2n}C_r$ is the greatest if and only if it is the coefficient of the medium term of expansion $(1+x)^{2n}$ (for example). Thus the greatest value of ${}^{2n}C_r$ is ${}^{2n}C_n$.

By the property ${}^{n}C_m = {}^{n-1}C_{m-1} + {}^{n-1}C_m$ we have

$${}^{100}C_r = {}^{99}C_{r-1}\,{}^{1}C_0 + {}^{99}C_r\,{}^{1}C_1 = {}^{98}C_{r-2}\,{}^{2}C_0 + {}^{98}C_{r-1}\,{}^{2}C_1 + {}^{98}C_{r-1}\,{}^{2}C_2 = \cdots$$
$$= {}^{40}C_0\,{}^{60}C_r + {}^{40}C_1\,{}^{60}C_{r-1} + \dots + {}^{40}C_{40}\,{}^{60}C_{r-40}.$$

Then by the statement-II the greatest value of ${}^{100}C_r = {}^{2\cdot 50}C_r$ occurs at $r = 50$.

4: **Statement-I.**

If $x = {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \dots + {}^{2n}C_{n-1}$ then $\frac{x+1}{2n+1}$ is integer.

Because

Statement-II. ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ and ${}^{n}C_r$ is divisible by n if n and r are co-prime

Solution.

$${}^{n}C_r + {}^{n}C_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{n!(n-r+1+r)}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

So as ${}^{n}C_n = 1$, ${}^{n}C_k = {}^{n}C_{n-k}$ and ${}^{n}C_0 + {}^{n}C_1 = {}^{n+1}C_1$, ${}^{n+1}C_1 + {}^{n+1}C_2 = {}^{n+2}C_2$ and so on, then

$$x + 1 = {}^{n}C_n + {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \dots + {}^{2n}C_{n-1}$$
$$= {}^{n}C_0 + {}^{n}C_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \dots + {}^{2n}C_{n+1}$$
$$= {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \dots + {}^{2n}C_{n+1} = {}^{n+2}C_2 + {}^{n+2}C_3 + \dots + {}^{2n}C_{n+1} = \cdots$$
$$= {}^{2n}C_n + {}^{2n}C_{n+1} = {}^{2n+1}C_{n+1}$$

So as $2n+1$ and $n+1$ are co-prime then ${}^{2n+1}C_{n+1}$ is divisible by $2n+1$ and $\frac{x+1}{2n+1}$ is integer.
