Answer on Question #56645 – Math – Combinatorics | Number Theory

4. If $(1+x+x^2+\cdots+x^p)^n = a_0 + a_1x + \cdots + a_{np}x^{np}$ then $a_1 + 2a_2 + 3a_3 + \cdots + np \cdot a_{np} =$

Solution

Let differentiate both sides of equality

$$(1+x+x^2+\cdots+x^p)^n = a_0 + a_1x + \cdots + a_{np}x^{np}$$

with respect to x:

$$\left(a_0 + a_1x + \cdots + a_{np}x^{np}\right)' = a_1 + 2a_2x + 3a_3x^2 + \cdots + np \cdot a_{np}x^{np-1}$$
$$= n(1+x+x^2+\cdots+x^p)^{n-1} \cdot (1+2x+3x^2+\cdots+px^{p-1}).$$

Then substituting for $x=1$ we obtain the desired expression

$$a_1 + 2a_2 + 3a_3 + \cdots + np \cdot a_{np}.$$

Thus,

$$a_1 + 2a_2 + 3a_3 + \cdots + np \cdot a_{np} = n(p+1)^{n-1} \cdot \frac{1+p}{2} \cdot p = \frac{n(p+1)^n}{2} \cdot p.$$

Answer: $\frac{n(p+1)^n}{2} \cdot p$.

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3. The term independent of x in the expansion of $\left[x^2 + \frac{1}{x}\right]^{15}$ is

Solution. Because $\left[x^2 + \frac{1}{x}\right]^{15} = \sum_{k=0}^{15} {}^{15}C_k (x^2)^k \left(\frac{1}{x}\right)^{15-k} = \sum_{k=0}^{15} {}^{15}C_k x^{3k-15}$ and $3k - 15 = 0$ for $k = 5$ we obtain that the term independent of x is

$${}^{15}C_5 = \frac{15!}{5!\cdot 10!} = \frac{15\cdot14\cdot13\cdot12\cdot11\cdot10!}{5\cdot4\cdot3\cdot2\cdot1\cdot10!} = \frac{15\cdot14\cdot13\cdot12\cdot11}{5\cdot4\cdot3\cdot2} = \frac{14\cdot13\cdot12\cdot11}{4\cdot2} = 7\cdot13\cdot3\cdot11 = 3003.$$

Answer: 3003.
