Answer on Question 56615 – Math – Combinatorics |Number Theory

We need the following properties of binomial coefficients

$$\boxed{C_k = C_k^n = \frac{n!}{k!(n-k)!}}$$

$$\boxed{C_k^n = C_{n-k}^n}$$

$$\boxed{\sum_{k=0}^{n} C_k^n = 2^n}$$

$$\boxed{\sum_{k=0}^{n} (-1)^k C_k^n = 0}$$

QUESTION 9

Prove that:

$$\sum_{r=1}^{n} \binom{n-1}{n-r} \binom{n}{r} = \binom{2n-1}{n-1}$$

SOLUTION

$$(1+x)^n = \sum_{k=0}^{n} C_k^n x^k$$

$$(1+x)^{n-1}(1+x)^n = (1+x)^{2n-1}$$

$$\left(\sum_{m=0}^{n-1} C_m^{n-1} x^m\right)\left(\sum_{l=0}^{n} C_l^n x^l\right) = \sum_{k=0}^{2n-1} C_k^{2n-1} x^k$$

$$\sum_{m=0}^{n-1}\sum_{l=0}^{n} C_l^n C_m^{n-1} x^{m+l} = \sum_{k=0}^{2n-1} C_k^{2n-1} x^k$$

$$m + l = k$$

$$\sum_{k=0}^{2n-1}\left(\sum_{l=0}^{k} C_l^n C_{k-l}^{n-1} x^k\right) = \sum_{k=0}^{2n-1} C_k^{2n-1} x^k$$

$$\sum_{l=0}^{k} C_l^n C_{k-l}^{n-1} = C_k^{2n-1}$$

$$k = n \to \sum_{l=0}^{n} C_l^n C_{n-l}^{n-1} = C_n^{2n-1} \equiv C_{n-1}^{2n-1}$$

ANSWER

$$\sum_{l=0}^{n} C_l^n C_{n-l}^{n-1} = C_{n-1}^{2n-1}$$

QUESTION 10
Prove that:

$$\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \dots + \frac{C_n}{n+2} = \frac{n * 2^{n+1} + 1}{(n+1)(n+2)}$$

SOLUTION

$$\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \dots + \frac{C_n}{n+2} = \sum_{k=0}^{n} C_k^n \frac{1}{k+2}$$

$$C_k^n \frac{1}{k+2} = \frac{n!}{k!(n-k)!}\frac{1}{k+2} = \frac{n!}{k!(k+2)(n-k)!} = \frac{n!(k+1)}{k!(k+1)(k+2)(n-k)!} =$$

$$= \frac{n!(k+1)}{(k+2)!(n-k)!} = \frac{n!(k+1)}{(k+2)!(n+2-(k+2))!}\frac{(n+1)(n+2)}{(n+1)(n+2)} =$$

$$= \frac{k+1}{(n+1)(n+2)}\frac{(n+2)!}{(k+2)!(n+2-(k+2))!} = \frac{k+1}{(n+1)(n+2)}C_{k+2}^{n+2}$$

$$\sum_{k=0}^{n} C_k^n \frac{1}{k+2} = \sum_{k=0}^{n} \frac{k+1}{(n+1)(n+2)}C_{k+2}^{n+2} = \begin{pmatrix} k+2=t \\ k=0 \rightarrow\ t=2 \\ k=n \rightarrow\ t=n+2 \end{pmatrix} =$$

$$= \sum_{t=2}^{n+2} \frac{t-1}{(n+1)(n+2)}C_t^{n+2} = \frac{1}{(n+1)(n+2)}\left(\sum_{t=2}^{n+2} tC_t^{n+2} - \sum_{t=2}^{n+2} C_t^{n+2}\right)$$

$$1)\ \sum_{t=2}^{n+2} tC_t^{n+2} = \sum_{t=2}^{n+2} \frac{t(n+2)!}{t!(n+2-t)!} = \sum_{t=2}^{n+2} \frac{(n+2)!}{(t-1)!(n+2-(t-1)-1)!} =$$

$$= \sum_{t=2}^{n+2} \frac{(n+2)!}{(t-1)!(n+1-(t-1))!} = \sum_{t=2}^{n+2} \frac{(n+1)!(n+2)}{(t-1)!(n+1-(t-1))!} = \sum_{t=2}^{n+2} (n+2)C_{t-1}^{n+1} =$$

$$= \begin{pmatrix} t-1=l \\ t=2 \rightarrow\ l=1 \\ t=n+2 \rightarrow\ l=n+1 \end{pmatrix} = (n+2)\sum_{l=1}^{n+1} C_l^{n+1} = (n+2)\left(\sum_{l=0}^{n+1} C_l^{n+1} - C_0^{n+1}\right) =$$

$$= (n+2)\left(\underbrace{\sum_{l=0}^{n+1} C_l^{n+1}}_{2^{n+1}} - \underbrace{C_0^{n+1}}_{1}\right) = (n+2)(2^{n+1}-1)$$

$$\boxed{\sum_{t=2}^{n+2} tC_t^{n+2} = (n+2)(2^{n+1}-1)}$$

$$2)\ \sum_{t=2}^{n+2} C_t^{n+2} = \sum_{t=0}^{n+2} C_t^{n+2} - C_0^{n+2} - C_1^{n+2} = \underbrace{\sum_{t=0}^{n+2} C_t^{n+2}}_{2^{n+2}} - \frac{(n+2)!}{0!(n+2-0)!} - \frac{(n+2)!}{1!(n+2-1)!} =$$

$$= 2^{n+2} - \frac{(n+2)!}{(n+2)!} - \frac{(n+2)!}{(n+1)!} = 2^{n+2} - 1 - (n+2)$$

$$\boxed{\sum_{t=2}^{n+2} C_t^{n+2} = 2^{n+2} - n - 3}$$

$$\sum_{k=0}^{n} C_k^n \frac{1}{k+2} = \frac{1}{(n+1)(n+2)}\left(\sum_{t=2}^{n+2} tC_t^{n+2} - \sum_{t=2}^{n+2} C_t^{n+2}\right) =$$

$$\frac{(n+2)(2^{n+1}-1) - (2^{n+2}-n-3)}{(n+1)(n+2)} = \frac{n2^{n+1} - n + 2^{n+2} - 2 - 2^{n+2} + n + 3}{(n+1)(n+2)} =$$

$$= \frac{n2^{n+1}+1}{(n+1)(n+2)}$$

$$\boxed{\sum_{k=0}^{n} C_k^n \frac{1}{k+2} = \frac{n2^{n+1}+1}{(n+1)(n+2)}}$$

ANSWER

$$\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \frac{C_3}{5} + \cdots + \frac{C_n}{n+2} = \sum_{k=0}^{n} C_k^n \frac{1}{k+2} = \frac{n2^{n+1}+1}{(n+1)(n+2)}$$

QUESTION 11

Prove that:

$$\frac{1}{2}C_1^n - \frac{2}{3}C_2^n + \frac{3}{4}C_3^n - \frac{4}{5}C_4^n + \cdots + \frac{(-1)^{n+1}n}{n+1}C_n^n = \frac{1}{n+1}$$

SOLUTION

$$\frac{1}{2}C_1^n - \frac{2}{3}C_2^n + \frac{3}{4}C_3^n - \frac{4}{5}C_4^n + \cdots + \frac{(-1)^{n+1}n}{n+1}C_n^n = \sum_{k=0}^{n} C_k^n \frac{(-1)^{k+1}k}{k+1}$$

$$\frac{1}{k+1}C_k^n = \frac{1}{k+1}\frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n+1-(k+1))!} =$$

$$= \frac{1}{n+1}\frac{n!(n+1)}{(k+1)!(n+1-(k+1))!} = \frac{1}{n+1}C_{k+1}^{n+1}$$

$$\sum_{k=0}^{n} C_k^n \frac{(-1)^{k+1}k}{k+1} = \sum_{k=0}^{n} C_{k+1}^{n+1}\frac{(-1)^{k+1}k}{n+1} = \left(\begin{array}{l} k+1=t \\ k=0 \to\ t=1 \\ k=n \to\ t=n+1 \end{array}\right) =$$

$$= \frac{1}{n+1}\sum_{t=1}^{n+1} C_t^{n+1}(-1)^t(t-1) = \frac{1}{n+1}\left(\sum_{t=1}^{n+1} t\, C_t^{n+1}(-1)^t - \sum_{t=1}^{n+1} C_t^{n+1}(-1)^t\right)$$

$$1)\ \sum_{t=1}^{n+1} t\, C_t^{n+1}(-1)^t = \sum_{t=1}^{n+1} t\frac{(n+1)!}{t!(n+1-t)!}(-1)^t = \sum_{t=1}^{n+1} \frac{(n+1)!}{(t-1)!(n+1-(t-1)-1)!}(-1)^t =$$

$$= \sum_{t=1}^{n+1} \frac{(n+1)n!}{(t-1)!(n-(t-1))!}(-1)^t = \sum_{t=1}^{n+1}(n+1)C_{t-1}^n(-1)^t = \left(\begin{array}{l} t-1=l \\ t=1 \to\ t=0 \\ t=n+1 \to\ l=n \end{array}\right) =$$

$$= \sum_{l=0}^{n}(n+1)C_l^n(-1)^{l+1} = -(n+1)\sum_{l=0}^{n} C_l^n(-1)^l$$

$$\boxed{\sum_{t=1}^{n+1} t\, C_t^{n+1}(-1)^t = -(n+1)\sum_{l=0}^{n} C_l^n(-1)^l}$$

$$2)\sum_{t=1}^{n+1} C_t^{n+1}(-1)^t = \sum_{t=0}^{n+1} C_t^{n+1}(-1)^t - C_0^{n+1}(-1)^0 =$$

$$= \sum_{t=0}^{n+1} C_t^{n+1}(-1)^t - \frac{(n+1)!}{0!(n+1-0)!}(-1)^0 = \sum_{t=0}^{n+1} C_t^{n+1}(-1)^t - 1$$

$$\boxed{\sum_{t=1}^{n+1} C_t^{n+1}(-1)^t = \sum_{t=0}^{n+1} C_t^{n+1}(-1)^t - 1}$$

$$\sum_{k=0}^{n} C_k^n \frac{(-1)^{k+1}k}{k+1} = \frac{1}{n+1}\left(\sum_{t=1}^{n+1} t\, C_t^{n+1}(-1)^t - \sum_{t=1}^{n+1} C_t^{n+1}(-1)^t\right) =$$

$$= \frac{1}{n+1}\left(-(n+1)\sum_{l=0}^{n} C_l^n(-1)^l - \left(\sum_{t=0}^{n+1} C_t^{n+1}(-1)^t - 1\right)\right) =$$

$$= \frac{1}{n+1}\left(-(n+1)\underbrace{\sum_{l=0}^{n} C_l^n(-1)^l}_{0} - \underbrace{\sum_{t=0}^{n+1} C_t^{n+1}(-1)^t}_{0} + 1\right) = \frac{1}{n+1}$$

ANSWER

$$\frac{1}{2}C_1^n - \frac{2}{3}C_2^n + \frac{3}{4}C_3^n - \frac{4}{5}C_4^n + \cdots + \frac{(-1)^{n+1}n}{n+1}C_n^n = \sum_{k=0}^{n} C_k^n \frac{(-1)^{k+1}k}{k+1} = \frac{1}{n+1}$$

QUESTION 12

Prove that:

$$\left(C_1^{2n}\right)^2 + 2\left(C_2^{2n}\right)^2 + 3\left(C_3^{2n}\right)^2 + \cdots + 2n\left(C_{2n}^{2n}\right)^2 = \frac{(4n-1)!}{[(2n-1)!]^2}$$

SOLUTION

$$\left(C_1^{2n}\right)^2 + 2\left(C_2^{2n}\right)^2 + 3\left(C_3^{2n}\right)^2 + \cdots + 2n\left(C_{2n}^{2n}\right)^2 = \sum_{k=1}^{2n} k\ \left(C_k^{2n}\right)^2 =$$

$$= \sum_{k=1}^{2n} k\, C_k^{2n} * C_k^{2n} = \sum_{k=1}^{2n} k\, \frac{(2n)!}{k!(2n-k)!} * C_k^{2n} = \sum_{k=1}^{2n} \frac{(2n)!}{(k-1)!(2n-1-(k-1))!} * C_k^{2n} =$$

$$= \sum_{k=1}^{2n} \frac{2n*(2n-1)!}{(k-1)!(2n-1-(k-1))!} * C_k^{2n} = \sum_{k=1}^{2n} 2n\, C_{2n-1}^{k-1} * C_k^{2n} =$$

$$\left(\begin{array}{l}\text{using question 9}\\ \sum_{r=1}^{n} C_{n-r}^{n-1} C_r^n = C_{n-1}^{2n-1}\end{array}\right)$$

$$= 2n\, C_{2n-1}^{2*2n-1} = 2n\frac{(4n-1)!}{(2n-1)!(4n-1-(2n-1))!} = 2n\frac{(4n-1)!}{(2n-1)!(2n)!} = \frac{(4n-1)!}{[(2n-1)!]^2}$$

ANSWER

$$\left(C_1^{2n}\right)^2 + 2\left(C_2^{2n}\right)^2 + 3\left(C_3^{2n}\right)^2 + \cdots + 2n\left(C_{2n}^{2n}\right)^2 = \frac{(4n-1)!}{[(2n-1)!]^2}$$