Answer on Question #56613 – Math – Combinatorics | Number Theory

1. *The coefficient of* x^r *in the expression*
$(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+\cdots+(x+1)^{n-1}$ *is*
(A) ${}^nC_r(2^r-1)$ (B) ${}^nC_r(2^{n-r}-1)$ (C) ${}^nC_r(2^r+1)$ (D) ${}^nC_r(2^{n-r}+1)$

Solution

Let
$(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+\cdots+(x+1)^{n-1}=\sum_{r=0}^{n-1}a_r x^r$.
According to the formula
$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$,
multiply both sides of

$(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+\cdots+(x+1)^{n-1}=\sum_{r=0}^{n-1}a_r x^r$.

by $\big((x+2)-(x+1)\big)=1$, then

$$\big((x+2)-(x+1)\big)\cdot\big((x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+\cdots+(x+1)^{n-1}\big)=$$
$$=(x+2)^n-(x+1)^n=\big((x+2)-(x+1)\big)\sum_{r=0}^{n-1}a_r x^r,$$

hence
$(x+2)^n-(x+1)^n=\sum_{r=0}^{n-1}a_r x^r$.
Applying

$$(x+y)^n=\sum_{j=0}^{n}C_n^j x^j y^{n-j}=\sum_{j=0}^{n}C_n^j x^{n-j} y^j$$

to both terms of $(x+2)^n-(x+1)^n$, come to the following expression:

$\sum_{r=0}^{n-1}2^{n-r}\binom{n}{r}x^r-\sum_{r=0}^{n-1}\binom{n}{r}x^r=\sum_{r=0}^{n-1}a_r x^r$,
$\sum_{r=0}^{n-1}(2^{n-r}-1)\binom{n}{r}x^r=\sum_{r=0}^{n-1}a_r x^r$.

Equating similar terms in both sides of the last equality gives $(2^{n-r}-1)\binom{n}{r}=a_r$.

Answer: (B) $C_n^r(2^{n-r}-1)$.

2. *If* $(1+x+x^2)^{25}=a_0+a_1x+a_2x^2+\cdots+a_{50}x^{50}$*, then* $a_0+a_2+a_4+\cdots+a_{50}$ *is*
(A) even (B) odd & of the form 3n (C) odd & of the form (3n-1) (D) odd & of the form (3n+1)

Solution

We believe that there is a mistake in the question. There should be

$$a_0+a_1+a_2+a_3+a_4+\cdots+a_{50}.$$

If we put $x=1$ in $(1+x+x^2)^{25}=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_{50}x^{50}$, then $(1+1+1^2)^{25}=a_0+a_1\cdot 1+a_2\cdot 1^2+a_3\cdot 1^3+a_4\cdot 1^4+\cdots+a_{50}\cdot 1^{50}$, that is,

$$a_0+a_1+a_2+a_3+a_4+\cdots+a_{50}=3^{25}$$

If we put $x=-1$ in $(1+x+x^2)^{25}=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_{50}x^{50}$, then $(1-1+1^2)^{25}=a_0-a_1+a_2\cdot 1^2-a_3\cdot 1^3+a_4\cdot 1^4+\cdots+a_{50}\cdot 1^{50}$, that is

$$a_0-a_1+a_2-a_3+a_4+\cdots+a_{50}=1^{25}=1.$$

Adding equalities from both cases gives

$$(a_0+a_1+a_2+a_3+a_4+\cdots+a_{50})+(a_0-a_1+a_2-a_3+a_4+\cdots+a_{50})=$$
$$=2(a_0+a_2+a_4+a_6+\cdots+a_{50})=1+3^{25},$$

hence

$a_0+a_2+a_4+a_6+\cdots+a_{50}=\frac{1+3^{25}}{2}$.

By multinomial theorem,

$$\left(x_1 + x_2 + \ldots + x_k\right)^n = \sum_{n1+n2+\ldots nk=n}^{n} P_n(n_1, n_2, \ldots, n_k) x_1^{n1} x_2^{n2} \ldots x_n^{nk}$$

If $\left(x_1 + x_2 + x_3\right)^n = a_0 x_1^n + \cdots$ then the sum of coefficient is odd and of the form 3n. Indeed,

$$(1 + x + x^2)^1 : a_0 + a_1 + a_2 = 3$$
$$(1 + x + x^2)^2 : a_0 + a_1 + a_2 + \cdots + a_4 = 3^2$$
$$(1 + x + x^2)^3 : a_0 + a_1 + a_2 + \cdots + a_6 = 3^3$$
$$(1 + x + x^2)^4 : a_0 + a_1 + a_2 + \cdots + a_8 = 3^4$$
$$\ldots$$
$$(1 + x + x^2)^{25} : a_0 + a_1 + a_2 + \cdots + a_{50} = 3^{25}$$

Answer: (B) odd & of the form 3n

3. *The coefficient of* x^4 *in the expression* $(1 - x + 2x^2)^{12}$ *is*
(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D) ${}^{12}C_3$+3·${}^{13}C_3$+${}^{14}C_4$

Solution

$$\left(x_1 + x_2 + \ldots + x_k\right)^n == \sum_{n1+n2+\ldots nk=n}^{n} P_n(n_1, n_2, \ldots, n_k) x_1^{n1} x_2^{n2} \ldots x_n^{nk}$$

$$1^{10} \text{ and } x^0 \text{ and } (2x^2)^2 \text{ then } P_{12}(10,0,2) = \frac{12!}{10!\,0!\,2!} = 264$$
$$1^9 \text{ and } x^2 \text{ and } (2x^2)^1 \text{ then } P_{12}(9,2,1) = \frac{12!}{9!\,1!\,2!} = 1320$$
$$1^8 \text{ and } x^4 \text{ and } (2x^2)^0 \text{ then } P_{12}(8,4,0) = \frac{12!}{8!\,0!\,4!} = 495$$

The coefficient of x^4 *in the expression* $(1 - x + 2x^2)^{12}$ *is*

$$P_n(10,0,2) + P_n(9,2,1) + P_n(8,4,0) = 264 + 1320 + 495 = 2079$$

On the other hand,

$${}^{12}C_3 + 3 \cdot {}^{13}C_3 + {}^{14}C_4 = 2079$$

Answer: (D) ${}^{12}C_3$+3·${}^{13}C_3$+${}^{14}C_4$.

4. *Let* $(1 + x^2)^2 (x + 1)^n = A_0 + A_1 x + \cdots$ *If* A_0, A_1, A_2 *are in A.P. then the value of n is*
(A) 2 (B) 3 (C) 5 (D) 7

Solution:

$$(1 + x^2)^2 (x + 1)^n = (1 + 2x^2 + x^4)(1 + C_n^{n-1} x + C_n^{n-2} x^2 + \cdots) =$$
$$= 1 + C_n^{n-1} x + (C_n^{n-2} + 2) x^2 + \cdots$$

For A.P.: $A_0 + A_2 = 2A_1$

$$1 + C_n^{n-2} + 2 = 2C_n^{n-1}$$
$$\frac{n!}{(n-2)!\,2!} + 3 = 2\frac{n!}{(n-1)!\,1!}$$
$$\frac{n(n-1)}{2} + 3 = 2n$$
$$n^2 - n + 6 - 4n = 0$$
$$n^2 - 5n + 6 = 0$$
$$n = 2 \text{ and } n = 3$$

Answer: (A) 2, (B) 3.