Answer on Question #56612 – Math – Combinatorics | Number Theory

8. If a be the sum of the odd terms and b be the sum of the even terms in the expansion of $(1+x)^n$, then $(1-x^2)^n$ is equal to

(A) a^2-b^2 (B) a^2+b^2 (C) b^2-a^2 (D) none

Solution

Because

$$(1-x^2)^n = (1+x)^n(1-x)^n,$$

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n = a+b,$$

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \cdots + (-1)^{n-1}\,{}^nC_{n-1}x^{n-1} + (-1)^n\,{}^nC_{n-1}x^n = a-b,$$

hold, then

$$(1-x^2)^n = (1+x)^n(1-x)^n = (a+b)(a-b) = a^2-b^2$$

Answer: (A): a^2-b^2.

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9. The sum of the coefficient of all the even powers of x in the expancion of $(2x^2-3x+1)^{11}$ is

(A) $2\cdot 6^{10}$ (B) $3\cdot 6^{10}$ (C) 6^{11} (D) none

Solution

If we denote $P(\mathrm{x}) = (2\mathrm{x}^2-3\mathrm{x}+1)^{11} = \sum_{k+l+m=0}^{11}\frac{11!}{k!l!m!}(2x^2)^k(-3x)^l1^m$, then $P(-\mathrm{x}) = (2\mathrm{x}^2+3\mathrm{x}+1)^{11}$.

These polynomials are different from each other only with respect to the sign of coefficients by the odd powers of x.

Then the expansion of

$$Q(x) = P(\mathrm{x}) - P(-x)$$

only consists of the odd powers of x, the sum of their coefficients is equal to

$$Q(1) = P(1) - P(-1) = -6^{11}$$

It follows that the sum of the coefficients of all odd powers of x in the expancion of $P(x)$ is equal to

$$\frac{1}{2}\cdot(-6^{11}).$$

The sum of all coefficients in the expansion of $P(x)$ is equal to $P(1)=0$. Thus, the sum of the coefficient of all the even powers of x is equal to $0-\frac{1}{2}\cdot(-6^{11}) = \frac{1}{2}\cdot 6^{11} = 3\cdot 6^{10}$.

Answer: (B): $3\cdot 6^{10}$.
