Answer on Question #56500 – Math – Algebra

6. The sequence is defined as follows

$$a_1 = \frac{1}{2}, \qquad a_{n+1} = a_n^2 + a_n,$$
$$S = \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_{100}+1}$$

then find $[S]$ ($[\cdot]$ is G.I. function).

Solution. Let $b_n = \frac{1}{a_n+1}$ and therefore $S = \sum_{n=1}^{100} b_n$. We have

$$b_1 = \frac{1}{a_1+1} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}, b_2 = \frac{1}{a_2+1} = \frac{1}{\frac{1}{2}+\frac{1}{4}+1} = \frac{4}{7}$$

and $S > b_1 + b_2 = \frac{2}{3} + \frac{4}{7} = \frac{26}{21} > 1$.

In different way,

$$b_1 = \frac{1}{a_1+1} = \frac{1}{1+\frac{1}{2}}, b_2 = \frac{1}{a_2+1} = \frac{1}{\frac{1}{2}+\frac{1}{4}+1} = \frac{1}{1+\frac{3}{4}} = \frac{1}{1+\frac{1}{1+\frac{1}{3}}},$$

$$b_3 = \frac{1}{a_3+1} = \frac{1}{\frac{3}{4}+\frac{9}{16}+1} = \frac{1}{2+\frac{5}{16}} = \frac{1}{2+\frac{1}{3+\frac{1}{5}}}, \ldots$$

It follows that $\lim\limits_{n\to\infty} \sum_{k=1}^{n} b_k = 2$ (by the properties of continued fractions), but $\sum_{n=1}^{100} b_n < 2$. Thus, $1 < S < 2$ and

$$[S] = 1.$$

Answer: (A): 1.

7: Consider the equation $(x+iy)^{2002} = x - iy$. If the number of ordered pairs (x, y) is *N* satisfying the given equation then find the sum of digits of *N*.

Solution. We rewrite the complex number in trigonometric form

$$x + iy = z = r(\cos t + i \sin t),$$

where $t = \operatorname{Arg} z = \arctan\frac{y}{x}$, $r = |z| = \sqrt{x^2+y^2}$. Then $x - iy = \bar{z} = r(\cos t - i \sin t)$, and

$$(x+iy)^{2002} = r^{2002}(\cos t + i \sin t)^{2002} = r^{2002}(\cos 2002t + i \sin 2002t) =$$
$$= r(\cos t - i \sin t)$$

or $r^{2001}(\cos 2002t + i \sin 2002t) = \cos t - i \sin t$. Because $|\cos(\cdot)| \le 1, |\sin(\cdot)| \le 1$ the number r is bound to be equal to 1, and we have the next equivalent system

$$\cos(2002t) = \cos t, \qquad \sin(2002t) = -\sin t$$

The last system has 1001 solutions, for the pairs (x, y) we have 2002 solutions and so on $N = 2002$.

Answer: (A): 4.

8: If in an equation $|x| + |y| + |z| = 10$, $x, y, z \in I$. Then number of solutions is

Solution. If $x, y, z \in \mathbb{Z}\backslash\{0\}$ then number of solutions is equal to

$$\overline{P}_9(2,7) \cdot \overline{A_2^3} = 36 \cdot 8 = 288$$

Let only one of the variables x, y, z, is equal to zero. Then number of solutions is

$$3 \cdot \overline{A_2^2} \cdot \overline{P}_9(1,8) = 12 \cdot 9 = 108.$$

Let two of the variables x, y, z, is equal to zero. Then number of solutions is

$$3 \cdot 2 = 6.$$

It follows that the general number of solutions of the given equation is

$$288 + 108 + 6 = 402.$$

Answer: (B) 402.

9: It $\cot^2 x + 8\cot x + 3 = 0, x \in [0, 2\pi]$. Then sum of all the solutions is?

Solution. We rewrite the given equation

$$t^2 + 8t + 3 = 0$$

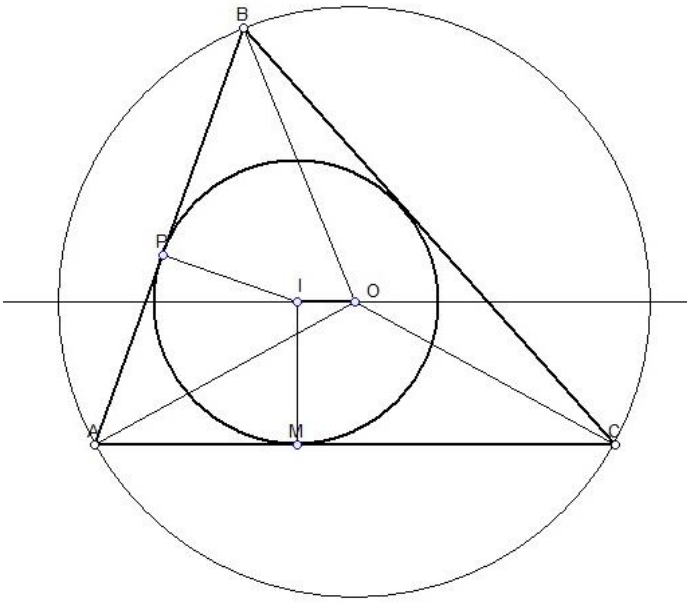
where $t = \cot x$. Then $t_1 = -4 - \sqrt{13}$ and $t_2 = -4 + \sqrt{13}$. It follows that the solutions of this equation, which belong to the segment $[0, 2\pi]$, are

$x = \pi - \operatorname{arccot}(4 + \sqrt{13})$ or $x = 2\pi - \operatorname{arccot}(4 + \sqrt{13})$ or $x = \pi - \operatorname{arccot}(4 - \sqrt{13})$ or $x = 2\pi - \operatorname{arccot}(4 - \sqrt{13})$

and the sum of all the solutions is $6\pi - 2\operatorname{arccot}(4 - \sqrt{13}) - 2\operatorname{arccot}(4 + \sqrt{13})$.

Answer: (D) None of these.

10: In any $\triangle ABC$, line joining circumcentre and Incentre is parallel to AC then OI is equal to (R is circumradius of $\triangle ABC$)?



Solution. If $AO = BO = CO = R$ and $OI \parallel AC$ then $\angle IOA = \angle OAC = \angle CAO$.

By the law of sines we have that

$$\frac{IO}{\sin \angle ICO} = \frac{CO}{\sin \frac{C}{2}}$$

and

$$\frac{IO}{\sin \angle IAO} = \frac{AO}{\sin \frac{A}{2}}$$

Because $\angle IAO = \frac{A}{2} - \angle IOA$ and $\angle ICO = \angle IOA - \frac{C}{2}$, then

$$\frac{IO}{\sin\left(\angle IOA - \frac{C}{2}\right)} = \frac{R}{\sin \frac{C}{2}},$$

$$\frac{IO}{\sin(\frac{A}{2} - \angle IOA)} = \frac{R}{\sin \frac{A}{2}},$$

It follows that $\frac{\sin(\frac{A}{2} - \angle IOA)}{\sin \frac{A}{2}} = \frac{\sin(\angle IOA - \sin \frac{C}{2})}{\sin \frac{C}{2}}$

and by the tangent law we obtain that $\frac{\sin(\frac{A}{2} - \angle IOA)}{\sin \frac{A}{2}} = \tan\left(\frac{A}{2} - \frac{C}{2}\right)$.Thus, $IO = R\left|\tan\left(\frac{A-C}{2}\right)\right|$.

Answer: (A) $R\left|\tan\left(\frac{A-C}{2}\right)\right|$.
