Answer on Question #56218 – Math – Linear Algebra

Question

If Ax =tx
Where t = 2 2 -2
1 3 1
1 2 1

Determine the eigen values of the matrix A, and an eigen vector corresponding to each eigen value.

If t = 4

Solution

Given a 3×3 matrix A, its eigenvector is a nonzero vector $x \in \mathbb{R}^3$ such that there exists $\lambda \in \mathbb{R}$ with the property $Ax = \lambda x$, which means that when we act on x by the linear operator determined by A, the image is a vector parallel to x. The scalars λ for which there exist $x \in \mathbb{R}^3$ that satisfy the equation $Ax = \lambda x$ are called eigenvalues of A.

To find eigenvalues of A, one has to solve the equation

$\det(A - \lambda E) = 0$ for λ, where $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In our case, if $A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}$, the equation takes the form

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0,$$

or, after the expansion,

$$-\lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0.$$

First look for rational roots among numbers $\{-1,1,2,-1,4,-4\}$ (recall that if $x_0 = \frac{m}{n}$ is a rational root of the polynomial $a_n x^n + \cdots + a_0$ then m divides a_0 and n divides a_n).

We see that $\lambda = 4$ and $\lambda = 1$ satisfy the equation.

In order to find the last root, divide $-\lambda^3 + 6\lambda^2 + 9\lambda + 4$ by $(\lambda - 4)(\lambda - 1) = \lambda^2 - 5\lambda + 4$. We obtain that

$$-\lambda^3 + 6\lambda^2 + 9\lambda + 4 = (\lambda - 4)(\lambda - 1)(\lambda - 1),$$

hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 1$ (the last root is of multiplicity 2).

To find eigenvalues corresponding to $\lambda_1 = 4$, solve the matrix equation

$Ax = 4x$ or $\begin{pmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

an equivalent system with triangular matrix is $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

from where

$$x_1 - x_2 + x_3 = 0$$

$$3x_2 = 4x_3$$

If $3x_2 = 4x_3$, then $x_2 = \frac{4}{3}x_3$.

Next,
if $x_1 - x_2 + x_3 = 0$, then $x_1 = x_2 - x_3 = \frac{4}{3}x_3 - x_3 = \frac{1}{3}x_3$.

We get the general solution (the linear subspace of eigenvectors corresponding to $\lambda_1 = 4$) of the form $\left(\frac{1}{3}x_3, \frac{4}{3}x_3, x_3\right)$, $x_3 \in \mathbb{R}$. Set $x_3 = 3$ to obtain vector $a = (1,4,3)$.

Similarly, for $\lambda_2 = 1$ we solve the system

$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

which is equivalent to

$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

from where

$$x_1 + 2x_2 - 2x_3 = 0$$

$$x_3 = 0$$

If $x_3 = 0$, then $x_1 + 2x_2 - 2x_3 = 0$ gives $x_1 + 2x_2 = 0$, hence $x_1 = -2x_2$.

The general solution is $(-2x_2, x_2, 0), x_2 \in \mathbb{R}$. Set $x_2 = 1$ to obtain vector $b = (-2,1,0)$.

Answer: $\lambda_1 = 4$ $a = (1,4,3)$, $\lambda_2 = 1$, $b = (-2,1,0)$.