TASK№1

If $f(x)=\sqrt{x}$ and $\phi(x)=\dfrac{1}{\sqrt{x}}$ in(a,b), then verify cauchy mean value theorem.

SOLUTION

First of all point to the possible values of a and b. Since the task is to verify the Cauchy theorem about average for the functions $f(x)=\sqrt{x}$ and $\phi(x)=\dfrac{1}{\sqrt{x}}$ that are defined for $\forall x>0$, so that $a>0$ and $b>0$. Without loss of generality we can assume that $a<b$

Recall the Cauchy's theorem about average:

Cauchy's mean value theorem, also known as the extended mean value theorem, is a generalization of the mean value theorem. It states: If functions $f(x)$ and $\phi(x)$ are both continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b), then there exists some $c\in(a,b)$, such that

$$\frac{f'(c)}{\phi'(c)}=\frac{f(a)-f(b)}{\phi(a)-\phi(b)}$$

To be convinced of the truth of the theorem is necessary to solve the equation with respect to C and show that it is $c\in(a,b)$

$$\begin{pmatrix} f(x)=\sqrt{x}\longrightarrow f'(x)=\frac{1}{2\sqrt{x}} \\ \phi(x)=\dfrac{1}{\sqrt{x}}\longrightarrow \phi'(x)=-\dfrac{1}{2}\dfrac{1}{\sqrt{x^3}} \end{pmatrix}$$

$$\frac{f'(c)}{\phi'(c)}=\frac{f(a)-f(b)}{\phi(a)-\phi(b)}\longrightarrow \frac{\dfrac{1}{2\sqrt{c}}}{-\dfrac{1}{2}\dfrac{1}{\sqrt{c^3}}}=\frac{\sqrt{a}-\sqrt{b}}{\dfrac{1}{\sqrt{a}}-\dfrac{1}{\sqrt{b}}}$$

$$-c=\frac{\sqrt{a}-\sqrt{b}}{\dfrac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}}}\longrightarrow c=\sqrt{ab}$$

We show that $c>a$

$$ab>a^2\longrightarrow a(b-a)>0-\text{because we have agreed that } a>0 \text{ and } a<b$$

We show that $c<b$

$$ab<b^2\longrightarrow b(a-b)<0-\text{because we have agreed that } b>0 \text{ and } a<b$$

We see that the found value c really lies in the interval (a,b)

Cauchy's mean value theorem verified.