Question #54770, Math / Calculus

Four bugs are placed at the four corners of a square with side length a. The bugs crawl counter-clockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the centre of the square along spiral paths.

a) Find the polar equation of a bug's path assuming the origin (pole) is at the centre of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path).

b) Find the distance travelled by a bug by the time it meets the other bugs at the centre.

Answer.

(a) The polar coordinates of the bug in the upper right-hand corner: $(\boldsymbol{r}, \boldsymbol{\theta})$.

The polar coordinates of the bug in the upper left-hand corner: $\left(\boldsymbol{r}, \boldsymbol{\theta}+\frac{\pi}{2}\right)$.

The Cartesian coordinates for these two bugs: $(\boldsymbol{rcos\theta}, \boldsymbol{rsin\theta})$ **and**

$$\left(\boldsymbol{rcos}\left(\boldsymbol{\theta}+\frac{\pi}{2}\right), \boldsymbol{rsin}\left(\boldsymbol{\theta}+\frac{\pi}{2}\right)\right)=(-\boldsymbol{rsin\theta}, \boldsymbol{rcos\theta}).$$

The slope of the line connecting these two points:

$$\boldsymbol{m}=\frac{\boldsymbol{rcos\theta}-\boldsymbol{rsin\theta}}{-\boldsymbol{rsin\theta}-\boldsymbol{rcos\theta}}=\frac{\boldsymbol{tan\theta}-\mathbf{1}}{\boldsymbol{tan\theta}+\mathbf{1}}$$

The motion of 1-st bug is precisely in the direction of the 2-nd bug. In other words, $\frac{dy}{dx}=\boldsymbol{m}$, **where** $\frac{dy}{dx}$ **is evaluated at the position of the 1-rst bug when it lies at coordinates** $(\boldsymbol{r}, \boldsymbol{\theta})$. **). We have:**

$$\frac{dy}{dx}=\frac{\frac{dr}{d\theta}sin\theta+rcos\theta}{\frac{dr}{d\theta}cos\theta-rsin\theta}=\boldsymbol{m} \rightarrow \frac{\frac{dr}{d\theta}sin\theta+rcos\theta}{\frac{dr}{d\theta}cos\theta-rsin\theta}=\frac{tan\theta-1}{tan\theta+1} \rightarrow$$

$$\rightarrow\left(\frac{dr}{d\theta}\boldsymbol{sin\theta}+\boldsymbol{rcos\theta}\right)(\boldsymbol{tan\theta}+\mathbf{1})=\left(\frac{dr}{d\theta}\boldsymbol{cos\theta}-\boldsymbol{rsin\theta}\right)(\boldsymbol{tan\theta}-\mathbf{1}) \rightarrow$$

$$\rightarrow \quad \frac{dr}{d\theta}(tan^2\theta + 1) = -r(tan^2\theta + 1) \quad \rightarrow \frac{dr}{d\theta} = -r\,.$$

This equation has the solution $r(\theta) = ce^{-\theta}$**.**

We know that $r\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}$ **(half the length of the square's diagonal).**

So $\frac{a}{\sqrt{2}} = ce^{-\frac{\pi}{4}} \quad \rightarrow \quad c = \frac{a\sqrt{2}}{2}e^{\frac{\pi}{4}}$ **and finally**

$$r = \frac{a\sqrt{2}}{2}e^{\frac{\pi}{4}-\theta}\,.$$

(b) The distance traveled by the bug is

$$L = \int_{\frac{\pi}{4}}^{\infty}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\frac{\pi}{4}}^{\infty}\sqrt{\frac{a^2}{2}e^{\frac{\pi}{2}-2\theta} + \frac{a^2}{2}e^{\frac{\pi}{2}-2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\infty} ae^{\frac{\pi}{4}-\theta} =$$

$$= -ae^{\frac{\pi}{4}-\theta}\Big|_{\theta=\frac{\pi}{4}}^{\theta=\infty} = -a(0-1) = a.$$