Answer on Question #54768 - Math – Calculus

Task. Determine the convergence or divergence of the sequence of partial sums whose n-th term is given by:

a) $u_n = \sum_{k=1}^{n}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$

b) $u_n = n^2 e^{-n}$

Solution.

a) Find the sum of n-th term

$$u_n = \sum_{k=1}^{n}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) + \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) =$$

$$= \frac{1}{3} - \frac{1}{2n+3}$$

Then we use a necessary condition for convergence of the sum. We check next:

$$\lim_{n\to\infty} u_n = 0$$

$$\lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{2n+3}\right) = \frac{1}{3} \neq 0$$

Answer. The sequence of partial sums with this n-th term is divergence.

b) The first step is to check the necessary condition.

$$\lim_{n\to\infty} u_n = 0$$

$$\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} = 0$$

We get this result, because exponent is growing faster than the power function. u_n is a positive terms our series. We can use the d'Alembert's ratio test.

The usual form of the test makes use of the limit

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$$

The ratio test states that:

if *L* < 1 then the series converges;

if *L* > 1 then the series does not converge;

if *L* = 1 or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

$$\lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \times \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{e\cdot n^2} = \lim_{n\to\infty}\left(\frac{n^2}{e\cdot n^2} + \frac{2n}{e\cdot n^2} + \frac{1}{e\cdot n^2}\right) = \frac{1}{e} + 0 + 0 = \frac{1}{e}$$

We have that $\frac{1}{e} < 1$. Looking at the d'Alembert's ratio test we get the answer.

Answer. The sequence of partial sums with this n-th term is convergence.