Answer on Question #54503 – Math – Linear Algebra

Solve the set of linear equations by matrix method and Gaussian elimination method:

$$\begin{cases} a + 2b + 3c = 5 \\ 3a - b + 2c = 8 \\ 4a - 6b - 4c = -2 \end{cases}.$$

Solve for *b*.

Solution

I. Gaussian elimination method

1. Construct the augmented matrix for the system:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 3 & -1 & 2 & 8 \\ 4 & -6 & -4 & -2 \end{array}\right)$$

   Column 1 contains coefficients beside a.

   Column 2 contains coefficients beside b.

   Column 3 contains coefficients beside c.

   Row 1 is aasociated with the first equation.

   Row 2 is associated with the second equation.

   Row 3 is associated with the third equation.

   The vertical line between the columns containing the coefficient and the column containing the constant term are there to show you that this is an augmented matrix.

2. We use elementary row operations to transform this matrix into a triangular one.

   Subtract the first row, multiplied by 3, from the second row, the result is in the second row.

   We keep the first and the third rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 4 & -6 & -4 & -2 \end{array}\right)$$

   Subtract the first row, multiplied by 4, from the third one, the result is in the third row.

   We keep the first and the second rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 0 & -14 & -16 & -22 \end{array}\right)$$

   Divide the third row by two. We keep the first and the second rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 0 & -7 & -8 & -11 \end{array}\right)$$

   Subtract the second row from the third one, result is in the third row.

We keep the first and the second rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & -1 & -4 \end{array}\right)$$

Divide the second row by (-7), divide the third row by (-1).

We keep the first row unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

Subtract the third row from the second one, result is in the second row.

We keep the first and the third rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

Subtract the second row, multiplied by 2, and the third row, multiplied by 3, from the first row.

We keep the second and the third rows unchanged.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

This is a triangular matrix. Its associated system is

$$\begin{cases} a = -1 \\ b = -3 \\ c = 4 \end{cases}$$

Thus, $b = -3$.

II. Matrix method

Let's rewrite the system in matrix form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & -6 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}.$$

Then the solution is given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & -6 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}.$$

Let's find the inverse matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & -6 & 4 \end{pmatrix}^{-1}$:

$$\left(\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 3 & -1 & 2 & 0 & 1 & 0\\ 4 & -6 & -4 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & -7 & -7 & -3 & 1 & 0\\ 0 & -14 & -16 & -4 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & 0\\ 0 & 1 & 1 & \frac{3}{7} & -\frac{1}{7} & 0\\ 0 & 0 & 1 & -1 & 1 & -\frac{1}{2}\end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 0 & 1 & \frac{8}{7} & -\frac{5}{7} & \frac{1}{2}\\ 0 & 1 & 1 & \frac{10}{7} & -\frac{8}{7} & \frac{1}{2}\\ 0 & 0 & 1 & -1 & 1 & -\frac{1}{2}\end{array}\right).$$

So we have that

$$\begin{pmatrix}1 & 2 & 3\\ 3 & -1 & 2\\ 4 & -6 & 4\end{pmatrix}^{-1} = \begin{pmatrix}\frac{8}{7} & -\frac{5}{7} & \frac{1}{2}\\ \frac{10}{7} & -\frac{8}{7} & \frac{1}{2}\\ -1 & 1 & -\frac{1}{2}\end{pmatrix}.$$

and

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}\frac{8}{7} & -\frac{5}{7} & \frac{1}{2}\\ \frac{10}{7} & -\frac{8}{7} & \frac{1}{2}\\ -1 & 1 & -\frac{1}{2}\end{pmatrix} \cdot \begin{pmatrix}5\\ 8\\ -2\end{pmatrix} = \begin{pmatrix}-1\\ -3\\ 4\end{pmatrix}.$$

Answer: $b = -3$.