Answer on Question #54161 – Math – Differential Equations

A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per litre of water enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution

Let $y(t)$ be the amount of salt (in kilograms) after t minutes. We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$. Note that $\frac{dy}{dt}$ is the rate of change of the amount of salt, so

$$\frac{dy}{dt} = (rate\ in) - (rate\ out)$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank. We have

$$(rate\ in) = \left(0.03\frac{kg}{L}\right)\left(25\frac{L}{min}\right) = 0.75\frac{kg}{min}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $\frac{y(t)}{5000}$ (measured in kilograms per liter). Since the brine flows out at a rate of $25\frac{L}{min}$, we have

$$(rate\ out) = \left(\frac{y(t)}{5000}\frac{kg}{L}\right)\left(25\frac{L}{min}\right) = \frac{y(t)}{200}\frac{kg}{min}$$

Thus, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Since $y(0) = 20$, we have $-\ln|130| = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln|130|$$

Therefore

$$|150 - y| = 130e^{-\frac{t}{200}}$$

Since $y(t)$ is continuous and $y(0) = 20$ and the right side is never 0, we deduce that $150 - y$ is always positive. Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-\frac{t}{200}}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-\frac{30}{200}} \approx 38.1\ kg.$$

Answer: $\mathbf{38.1\ kg}$.