Answer on Question #54158– Math – Calculus

Question

Evaluate the improper integral to show:

∫ between 0 and ∞ ,(x arctan(x))/((1 + x^2)^2) dx = π/8

Solution

Definition 1 (Improper integral):

An integral is an *improper integral* if either the interval of integration is not finite (improper integral of *type 1*) or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of *type 2*).

Definition 2 (Improper integral of type 1):

Improper integrals of type 1 are evaluated as follows: if $\int_a^t f(x)dx$ exists for all $t \geq a$, then we define

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx \tag{1}$$

provided the limit exists as a finite number. In this case, $\int_a^{\infty} f(x)dx$ is said to be *convergent* (or to *converge*). Otherwise, $\int_a^{\infty} f(x)dx$ is said to be *divergent* (or to *diverge*).

Show that

$$\int \frac{dx}{(1+x^2)^2} = \left| x = \tan(t), dx = \frac{dt}{cos^2 t},\ 1 + x^2 = 1 + tan^2(t) = \frac{1}{cos^2(t)} \right| = \int cos^4(t) \frac{dt}{cos^2(t)} =$$

$$= \int cos^2(t)dt = \int \frac{1+\cos(2t)}{2} dt = \frac{t}{2} + \frac{1}{4}\sin(2t) + C = \frac{t}{2} + \frac{1}{2} sint \cdot cost + C =$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{\tan(t)}{\sqrt{tan^2(t)+1}} \cdot \frac{1}{\sqrt{tan^2(t)+1}} + C = \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{\tan(t)}{tan^2(t)+1} + C =$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{x^2+1} + C = \frac{1}{2}\left(\arctan(x) + \frac{x}{x^2+1}\right) + C,$$

where C is an integration constant.

According to the statement of the problem we have

$$\int_0^{\infty} \frac{x\, arctan(x)}{(1+x^2)^2} dx = \frac{\pi}{8}. \tag{2}$$

As we see, the left side of the relation (2) is the improper integral of type 1. Using (1) we rewrite the improper integral (2) in the form

$$\int_0^{\infty} \frac{x\, arctan(x)}{(1+x^2)^2} dx = \lim_{t\to\infty} \int_0^t \frac{x\, arctan(x)}{(1+x^2)^2} dx. \tag{3}$$

Now, we take the integral and pass to limit:

$$\int \frac{x\, arctan(x)}{(1+x^2)^2}dx =$$

$$= \left\{\text{the integration by parts formula: } \int U(x)V'(x)dx = \mathrm{U(x)V(x)} - \int V(x)U'(x)dx\right\}$$

$$= \left\{\begin{matrix} U(x) = \arctan(x)\,, U'(x) = \dfrac{1}{1+x^2}, V'(x) = \dfrac{x}{(1+x^2)^2}, \\ V(x) = \displaystyle\int \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{d(1+x^2)}{(1+x^2)^2} = -\frac{1}{2(1+x^2)} \end{matrix}\right\}$$

$$= \arctan(x)\cdot\left(-\frac{1}{2(1+x^2)}\right) - \int\left(-\frac{1}{2(1+x^2)}\right)\cdot\frac{1}{1+x^2}dx$$

$$= -\frac{1}{2}\frac{\arctan(x)}{(1+x^2)} + \frac{1}{2}\int\frac{dx}{(1+x^2)^2} = -\frac{1}{2}\frac{\arctan(x)}{(1+x^2)} + \frac{1}{4}\left(\arctan(x) + \frac{x}{1+x^2}\right) + C$$

$$= \frac{(1+x^2)\cdot\arctan(x) + x - 2\arctan(x)}{4(1+x^2)} + C = \frac{(x^2-1)\cdot\arctan(x) + x}{4(1+x^2)} + C;$$

$$\int_0^\infty \frac{x\, arctan(x)}{(1+x^2)^2}dx = \lim_{t\to\infty}\left(\frac{(x^2-1)\cdot\arctan(x)+x}{4(1+x^2)}\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(\frac{(t^2-1)\cdot\arctan(t)+t}{4(1+t^2)} - \frac{(0^2-1)\cdot\arctan(0)+0}{4(1+0^2)}\right)$$

$$= \lim_{t\to\infty}\left(\frac{t^2\cdot\arctan(t)}{4(1+t^2)} - \frac{\arctan(t)}{4(1+t^2)} + \frac{t}{4(1+t^2)}\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{4t^2}\frac{t^2\cdot\arctan(t)}{\left(\frac{1}{t^2}+1\right)} - \frac{1}{4t^2}\frac{\arctan(t)}{\left(\frac{1}{t^2}+1\right)} + \frac{1}{4t}\frac{1}{\left(\frac{1}{t^2}+1\right)}\right)$$

$$= \frac{1}{4}\cdot\frac{\frac{\pi}{2}}{(0+1)} - 0\cdot\frac{\frac{\pi}{2}}{(0+1)} + 0\cdot\frac{1}{(0+1)} = \frac{\pi}{8}.$$

Therefore, the relation (2) is satisfied and the given improper integral $\int_0^\infty \frac{x\, arctan(x)}{(1+x^2)^2}dx$ is convergent.