Answer on Question #54156 – Math – Real Analysis

Question

Let C be a subset of $[0,1]$, the cantor set and let $f: [0,1] \to [0, \infty)$ be given by $f(x) = 0$ on C and $f(x) = n$ in each complementary interval of length 3^{-n}. Show that f is Lebesgue measurable and compute $\int_0^1 f(x)dx$.

Solution

$$f(x) = \begin{cases} 0 \ on \ C \\ n \ on \ [0,1] \setminus C \end{cases} = 0 \cdot \chi_C(x) + n \cdot \chi_{[0,1]\setminus C}(x) = \sum_{k=1}^{n} a_k \chi_{A_k}, \ x \in [0,1],$$

where $\chi_C(x) = \begin{cases} 1 \ if \ x \in C, \\ 0, \ \ if \ x \notin C, \end{cases}$ is a characteristic function of a set C,

$\chi_{[0,1]\setminus C}(x) = \begin{cases} 1 \ if \ x \in [0,1] \setminus C, \\ 0, \ \ if \ x \notin [0,1] \setminus C, \end{cases}$ is a characteristic function of a set $[0,1] \setminus C$.

The characteristic function of a set E is measurable if and only if E is measurable.

The cantor set C is a Borel set and hence measurable. The closed interval $[0,1]$ is a Borel set and hence measurable.

Set $[0,1] \setminus C$ is Borel and hence measurable as complement of a measurable set C.

Thus, functions $\chi_C(x)$, $\chi_{[0,1]\setminus C}(x)$ are Lebesgue measurable, because sets C and $[0,1] \setminus C$ are measurable.

By properties of measurable functions, functions $0 \cdot \chi_C(x)$ and $n \cdot \chi_{[0,1]\setminus C}(x)$ will be Lebesgue measurable, besides, function $f(x) = 0 \cdot \chi_C(x) + n \cdot \chi_{[0,1]\setminus C}(x)$ is Lebesgue measurable as the sum of two Lebesgue measurable functions.

We showed that function f is Lebesgue measurable.

Next, function $f(x) = 0 \cdot \chi_C(x) + n \cdot \chi_{[0,1]\setminus C}(x)$ is simple, therefore

$\int_0^1 f(x)dx = \sum_{k=1}^{n} a_k \, \mu(A_k)$=$0 \cdot \mu(C) + n \cdot \mu([0,1] \setminus C) = 0 + n = n$, because

$\mu(C) = 0, \mu([0,1]) = 1 - 0 = 1.$