Answer on Question #54155 – Math – Real Analysis

If *A* is Lebesgue measurable subset of $\mathbb{R}$ of positive measure and $0 < \delta < \lambda(A)$, then show that there exists a measurable subset *B* of *A* satisfying $\lambda(B) = \delta$.

Solution

Assume without loss of generality that λ is Lebesgue measure on $\mathbb{R}$ and $\lambda(A) = 1$.

Define the function $f: \mathbb{R} \to [0,1]$ by

$$f(x) = \lambda(A \cap (-\infty, x]),$$

where $x \in \mathbb{R}$. It is continuous by the following inequality

$$|f(x) - f(y)| \leq |x - y|,$$

where $y \in \mathbb{R}$.

Since $\lim\limits_{x\to-\infty} f(x) = 0$ and $\lim\limits_{x\to+\infty} f(x) = 1$, there is a point $x_0 \in \mathbb{R}$ such that $f(x_0) = \delta$,

where $0 < \delta < 1$, i.e. $0 < \delta < \lambda(A)$.

Put $B = A \cap (-\infty, x]$, hence *B* is a measurable subset of *A* satisfying $\lambda(B) = \delta$, which was to be demonstrated.