Answer to question 53107

State and prove second mean value theorem for integrals

Second integral mean-value theorem. If the real functions f and g are continuous and f monotonic on the interval $[a,b]$, then the equation

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx + f(b)\int_{\xi}^b g(x)dx \tag{1}$$

is true for a value $\xi \in [a,b]$.

Proof. We can suppose that $f(a) \neq f(b)$ since otherwise any value of ξ between a and b would do.

Let's first prove the auxiliary result, that if a function φ is continuous on an open interval I containing $[a,b]$ then

$$\lim_{h\to 0}\int_a^b \frac{\varphi(x+h)-\varphi(x)}{h}dx = \varphi(b)-\varphi(a). \tag{2}$$

In fact, when we take an antiderivative Φ of φ, then for every nonzero h the function

$$x \mapsto \frac{\Phi(x+h)-\Phi(x)}{h}$$

is an antiderivative of the integrand of (2) on the interval $[a,b]$. Thus we have

$$\int_a^b \frac{\varphi(x+h)-\varphi(x)}{h}dx = \frac{\Phi(b+h)-\Phi(x)}{h} - \frac{\Phi(a+h)-\Phi(x)}{h} \to \varphi(b)-\varphi(a) \quad as \quad h\to 0.$$

The given functions f and g can be extended on an open interval I containing $[a,b]$ such that they remain continuous and f monotonic. We take an antiderivative G of g and a nonzero number h having small absolute value. Then we can write the identity

$$\int_a^b \frac{f(x+h)G(x+h)-f(x)G(x)}{h}dx$$
$$= \int_a^b \frac{f(x+h)[G(x+h)-G(x)]}{h}dx - \int_a^b \frac{f(x+h)-f(x)}{h}G(x)dx. \tag{3}$$

By (2), the left hand side of (3) may be written

$$\int_a^b \frac{f(x+h)G(x+h)-f(x)G(x)}{h}dx = f(b)G(b)-f(a)G(a)+\varepsilon_1(h) \tag{4}$$

where $\varepsilon_1(h)\to 0$ as $h\to 0$. Further, the function

$$x \to \begin{cases} \dfrac{f(x+h)[G(x+h)-G(x)]}{h} & for\ h\neq 0 \\ f(x)g(x) & for\ h = 0 \end{cases}$$

is continuous in a rectangle $a\le x\le b,\ -\delta\le h\le\delta,$ whence we have

$$\int_a^b \frac{f(x+h)[G(x+h)-G(x)]}{h}dx = \int_a^b f(x)g(x)dx + \varepsilon_2(h) \qquad (5)$$

where $\varepsilon_2(h) \to 0$ as $h \to 0$. Because of the monotonicity of f, the expression $\frac{f(x+h)-f(x)}{h}$ does not change its sign when $a \le x \le b$. Then the usual integral mean value theorem guarantees for every h (sufficiently near 0) a number ξ_h of the interval $[a, b]$ such that

$$\int_a^b \frac{f(x+h)-f(x)}{h}G(x)dx = G(\xi_h)\int_a^b \frac{f(x+h)-f(x)}{h}dx$$

and the auxiliary result (2) allows to write this as

$$\int_a^b \frac{f(x+h)-f(x)}{h}G(x)dx = G(\xi h)[f(b)-f(a)+\varepsilon_3(h)] \qquad (6)$$

with $\varepsilon_3(h) \to 0$ as $h \to 0$. Now the equations (4), (5) and (6) imply

$$f(b)G(b)-f(a)G(a)+\varepsilon_1(h) = \int_a^b f(x)g(x)dx + \varepsilon 2(h) + G(\xi h)[f(b)-f(a)+\varepsilon 3(h)]. \qquad (7)$$

Because $f(b)-f(a) \ne 0$, the expression $G(\xi_h)$ has a limit L for $h \to 0$. By the continuity of G there must be a number ξ between a and b such that $G(\xi) = L$. Letting then h tend to 0 we thus get the limiting equation

$$f(b)G(b)-f(a)G(a) = \int_a^b f(x)g(x)dx + G(\xi)[f(b)-f(a)]$$

which finally gives

$$\int_a^b f(x)g(x)dx = f(a)[G(\xi)-G(a)] + f(b)[G(b)-G(\xi)] = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx$$

Q.E.D.