Answer on Question #52986 – Math– Complex Analysis

Question:

What is the square root of $\sqrt{2x + i(x^2 - 1)}$?

Explanation:

It should be noted that, as it may seem, a complicated way of calculation yields the simple algebraic form of solution. Moreover, if we square the received solution, then we get the initial expression under the square root.

Let`s compare the algebraic and the exponential form of solution. By definition, the exponential form of complex number is

$$Z = X + iY = \rho e^{i\varphi}, \ \ \rho = |Z| = \sqrt{X^2 + Y^2}, \qquad \varphi = ArgZ = arctan\left(\frac{Y}{X}\right) + 2\pi k, k = 0, \pm 1, \pm 2, \ldots$$

In our case

$$Z = 2x + i(x^2 - 1), \qquad X = 2x, Y = (x^2 - 1),$$

and we get

$$\sqrt{Z} = \left(\rho e^{i\varphi}\right)^{\frac{1}{2}} = \sqrt{\rho} e^{\frac{i\varphi}{2}},$$

$$\rho = \sqrt{(2x)^2 + (x^2 - 1)^2} = \sqrt{4x^2 + x^4 - 2x^2 + 1} = \sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2 + 1)^2},$$

$$\varphi = arctan\left(\frac{(x^2 - 1)}{2x}\right) + 2\pi k,$$

$$\sqrt{Z} = \sqrt[4]{(x^2 + 1)^2} e^{\frac{i}{2}\left(arctan\left(\frac{(x^2-1)}{2x}\right) + 2\pi k\right)} = \sqrt{x^2 + 1} e^{\frac{i}{2}\left(arctan\left(\frac{(x^2-1)}{2x}\right) + 2\pi k\right)}, k = 0, \pm 1, \pm 2, \ldots.$$

We can see the exponential form of solution is bulky and implicit, unlike the algebraic form

$$\sqrt{Z} = \pm \frac{1}{\sqrt{2}}\big((x + 1) + i(x - 1)\big).$$

Solution

Let’s put

$$\sqrt{2x + i(x^2 - 1)} = a + ib, \tag{1}$$

where $a, b \in \mathbb{R}$. Rewriting (1) in the more convenient form

$$2x + i(x^2 - 1) = (a + ib)^2, \Longrightarrow$$

$$2x + i(x^2 - 1) = (a^2 - b^2) + 2iab. \tag{2}$$

Comparing the left-hand and right-hand sides of equality (2), we get the following system of equations

$$\begin{cases} a^2 - b^2 = 2x, \\ 2ab = x^2 - 1. \end{cases} \qquad (3)$$

Let's solve this system. By performing elementary transformations of (3)

$$\begin{cases} a^2 - b^2 = 2x, \\ 2ab = x^2 - 1, \end{cases} \Rightarrow \begin{cases} a^2 - b^2 = 2x, \\ \boxed{a = \dfrac{x^2 - 1}{2b}}, \end{cases} \Rightarrow \left(\frac{x^2 - 1}{2b}\right)^2 - b^2 = 2x, \Rightarrow (x^2 - 1)^2 - 4b^4 = 8xb^2,$$

we receive a biquadratic equation

$$4b^4 + 8xb^2 - (x^2 - 1)^2 = 0. \qquad (4)$$

Its solutions are

$$b_{1,2} = \pm\frac{1}{\sqrt{2}}(x - 1), \;\; b_{3,4} = \pm\frac{i}{\sqrt{2}}(x + 1). \qquad (5)$$

Further, substituting (5) into $a = \frac{x^2-1}{2b}$, we have

$$a_{1,2} = \frac{x^2 - 1}{2\cdot\left(\pm\frac{1}{\sqrt{2}}(x-1)\right)} = \pm\frac{(x-1)(x+1)}{\sqrt{2}(x-1)} = \pm\frac{(x+1)}{\sqrt{2}}; \qquad (6a)$$

$$a_{3,4} = \frac{x^2 - 1}{2\cdot\left(\pm\frac{i}{\sqrt{2}}(x+1)\right)} = \pm\frac{(x-1)(x+1)}{i\sqrt{2}(x+1)} = \mp\frac{i(x-1)}{\sqrt{2}}; \qquad (6b)$$

Therefore, the substituting (5), (6a) and (6b) into (1) yields the following result:

$$\sqrt{2x + i(x^2 - 1)} = \begin{cases} \pm\dfrac{(x+1)}{\sqrt{2}} \pm \dfrac{i}{\sqrt{2}}(x - 1) = \pm\dfrac{1}{\sqrt{2}}\big((x + 1) + i(x - 1)\big); \\ \mp\dfrac{i(x-1)}{\sqrt{2}} \pm \dfrac{i^2}{\sqrt{2}}(x + 1) = \mp\dfrac{i(x-1)}{\sqrt{2}} \mp \dfrac{1}{\sqrt{2}}(x + 1) = \mp\dfrac{1}{\sqrt{2}}\big((x + 1) + i(x - 1)\big). \end{cases}$$

$$\boxed{\sqrt{2x + i(x^2 - 1)} = \pm\frac{1}{\sqrt{2}}\big((x + 1) + i(x - 1)\big)} \qquad (7)$$

Answer: $\sqrt{2x + i(x^2 - 1)} = \pm\frac{1}{\sqrt{2}}\big((x + 1) + i(x - 1)\big)$.