Answer on Question #52953 – Math – Abstract Algebra

We know √9 = ±3 but why √-1 ≠ ± i. It's only i why?

Solution

We have

$$-1 = e^{(\pi+2\pi k)i}, k = 0,1,2, \dots$$

Thus

$\sqrt{-1} = e^{\frac{(\pi+2\pi k)}{2}i} = e^{\left(\frac{\pi}{2}+\pi k\right)i}, k = 0,1.$

If $k = 0$, then

$$\sqrt{-1} = e^{\left(\frac{\pi}{2}\right)i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

If $k = 1$, then

$$\sqrt{-1} = e^{\left(\frac{\pi}{2}+\pi\right)i} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

Make sure that other values of k do not give new values of $\sqrt{-1}$.

If $k = 2l$, l is integer, then

$\sqrt{-1} = e^{\frac{(\pi+2\pi k)}{2}i} = e^{\frac{(\pi+2l\cdot 2\pi)}{2}i} = e^{\left(\frac{\pi}{2}+2\pi l\right)i} = cos\left(\frac{\pi}{2}+2\pi l\right) + isin\left(\frac{\pi}{2}+2\pi l\right) = cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right) = i.$

If $k = 2l + 1$, l is integer, then

$\sqrt{-1} = e^{\frac{(\pi+2\pi k)}{2}i} = e^{\frac{(\pi+2\pi(2l+1))}{2}i} = e^{\left(\frac{3\pi}{2}+2\pi l\right)i} = cos\left(\frac{3\pi}{2}+2\pi l\right) + isin\left(\frac{3\pi}{2}+2\pi l\right) =$

$= cos\left(\frac{3\pi}{2}\right) + isin\left(\frac{3\pi}{2}\right) = -i.$

Therefore

$$\sqrt{-1} = \pm i.$$