Answer on Question #52764 – Math – Calculus

Question

Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following parametric equations:

$$y = t^2\, e^{-t^2}$$

$$x = \tan t$$

Solution

Method 1

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(t^2\, e^{-t^2}\right)_t'}{(\tan t)_t'}$$

Calculate

$$\frac{dy}{dt} = y_t' = \left(t^2\, e^{-t^2}\right)' = 2te^{-t^2} - 2t^3 e^{-t^2} = 2te^{-t^2}(1 - t^2),$$

$$\frac{dx}{dt} = x_t' = (\tan t)' = \frac{1}{cos^2 t}.$$

So

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(t^2\, e^{-t^2}\right)_t'}{(\tan t)_t'} = \frac{2te^{-t^2}(1-t^2)}{\frac{1}{cos^2 t}} = 2te^{-t^2}(1 - t^2)cos^2 t = e^{-t^2}(2t - 2t^3)cos^2 t,$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{e^{-t^2}(2t-2t^3)cos^2 t}{\frac{1}{cos^2 t}} = e^{-t^2}(2t - 2t^3)cos^4 t,$$

Method 2

If $x = \tan t$, then $t = \arctan x$.

$$t_x' = (\arctan x)' = \frac{1}{1 + x^2}.$$

$$y_t' = \left(t^2\, e^{-t^2}\right)' = 2te^{-t^2} - 2t^3 e^{-t^2} = 2te^{-t^2}(1 - t^2).$$

So,

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=2te^{-t^2}(1-t^2)*\frac{1}{1+x^2}=2\arctan x*e^{-\arctan^2 x}(1-arctan^2x)*\frac{1}{1+x^2}$$

Next,

$$\frac{d^2y}{dx^2}=\left(2\arctan x*e^{-\arctan^2 x}(1-arctan^2x)*\frac{1}{1+x^2}\right)'_x$$

$$=\left(e^{-\arctan^2 x}(2\arctan x-2arctan^3x)*\frac{1}{1+x^2}\right)'_x$$

$$=\frac{\left(e^{-\arctan^2 x}(2\arctan x-2arctan^3x)\right)'_x(1+x^2)-e^{-\arctan^2 x}(2\arctan x-2arctan^3x)(1+x^2)'_x}{(x^2+1)^2}$$

$$=\frac{\left(e^{-\arctan^2 x}(2\arctan x-2arctan^3x)\right)'_x}{x^2+1}-\frac{2xe^{-\arctan^2 x}(2\arctan x-2arctan^3x)}{(x^2+1)^2}$$

$$=\frac{(2\arctan x-2arctan^3x)\left(e^{-\arctan^2 x}\right)'_x}{x^2+1}+\frac{e^{-\arctan^2 x}(2\arctan x-2arctan^3x)'_x}{x^2+1}$$
$$-\frac{2xe^{-\arctan^2 x}(2\arctan x-2arctan^3x)}{(x^2+1)^2}$$

$$=\frac{(2\arctan x-2arctan^3x)e^{-\arctan^2 x}(-2arctanx)}{x^2+1}*\frac{1}{x^2+1}$$
$$+\frac{e^{-\arctan^2 x}\left(\frac{2}{1+x^2}-\frac{3*2arctan^2x}{1+x^2}\right)}{x^2+1}-\frac{4xarctanx*e^{-\arctan^2 x}}{(x^2+1)^2}+\frac{4xarctan^3x*e^{-\arctan^2 x}}{(x^2+1)^2}$$

$$=\frac{e^{-\arctan^2 x}(-4arctan^2x+4arctan^4x+2-6arctan^2x-4xarctanx+4xarctan^3x)}{(x^2+1)^2}$$

$$=\frac{2e^{-\arctan^2 x}(2arctan^4x-5arctan^2x-2xarctanx+2xarctan^3x+1)}{(x^2+1)^2}$$

Answer: $\frac{dy}{dx}=2\arctan x\, e^{-\arctan^2 x}(1-\arctan x^2)$.

$$\frac{d^2y}{dx^2}=\frac{2e^{-\arctan^2 x}(2arctan^4x-5arctan^2x-2xarctanx+2xarctan^3x+1)}{(x^2+1)^2}$$