Answer on Question #52706 – Math – Differential Calculus | Equations

Solve the following by Bernoulli equation?

dy/dx + xy/(1-x^2) = x y^1/2

Solution

$y' + \frac{x}{1-x^2} y = x\sqrt{y}$

Function $y = 0$ is the solution of the initial equation. Let $y \neq 0$ and divide both sides by $\sqrt{y}$:

$$\frac{y'}{\sqrt{y}} + \frac{x}{1-x^2}\sqrt{y} = x$$

Let $z = \sqrt{y} \rightarrow z' = \frac{y'}{2\sqrt{y}}$ so we have:

$2z' + \frac{x}{1-x^2} z = x$ **is a nonhomogeneous equation.**

Method 1

Let $z = uv$, $2u'v + 2uv' + \frac{x}{1-x^2} uv = x$,

$$\begin{cases} 2u'v + \dfrac{x}{1-x^2} uv = 0 \\ 2uv' = x \end{cases}$$

Let $v \neq 0$, divide the first equation $2u'v + \frac{x}{1-x^2} uv = 0$ of system by v and solve for u:

$$2u' + \frac{x}{1-x^2} u = 0$$

Solution of the separated homogeneous equation

$$2u' + \frac{x}{1-x^2} u = 0$$

$2\frac{du}{dx} = -\frac{x}{1-x^2}u$ is separated equation

$$2\int\frac{du}{u} = -\int\frac{xdx}{1-x^2} \rightarrow 2lnu = \frac{1}{2}ln(c(1-x^2)) \rightarrow u = c\sqrt[4]{1-x^2}$$

Next, take $c = 1$ and substitute $u = \sqrt[4]{1-x^2}$ into the second equation

$2uv' = x$ of the system and solve for v:

$2uv' = x$

$2\sqrt[4]{1-x^2}v' = x$

$2v' = \frac{x}{\sqrt[4]{1-x^2}}$

$$2v = \int\frac{x}{\sqrt[4]{1-x^2}}dx + c_1$$

$$2v = -\frac{1}{2}\int(1-x^2)^{-\frac{1}{4}}(-2x)\,dx + c_1$$

$$2v = -\frac{1}{2}\frac{(1-x^2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + c_1$$

$$2v = -\frac{2}{3}\sqrt[4]{(1-x^2)^3} + c_1$$

$v = -\frac{1}{3}\sqrt[4]{(1-x^2)^3} + c, \quad c = \frac{c_1}{2}$

Recall the substitution

$$z = uv = \sqrt[4]{1-x^2}\left(-\frac{1}{3}\sqrt[4]{(1-x^2)^3} + c\right) = -\frac{1-x^2}{3} + c\sqrt[4]{1-x^2} =$$

$$= \frac{x^2-1}{3} + c\sqrt[4]{1-x^2}$$

A particular solution of the nonhomogeneous equation :

$z_p = \frac{1}{3}(x^2 - 1).$

A general solution of the homogeneous equation :

$$z_c = c\sqrt[4]{1 - x^2}$$

So the general solution of the nonhomogeneous equation is

$z = c\sqrt[4]{1 - x^2} - \frac{1}{3}(\,1 - x^2)$

Recall the substitution $z = \sqrt{y}$ **and finally obtain that**

$\sqrt{y} = c\sqrt[4]{1 - x^2} - \frac{1}{3}(\,1 - x^2),$

where *c* is an arbitrary real constant

or $y = c\sqrt{1 - x^2} - \frac{2c}{3}(1 - x^2)\sqrt[4]{1 - x^2} + \frac{1}{9}(1 - x^2)^2$

Method 2

$2z' + \frac{x}{1-x^2}z = x$ **is a nonhomogeneous equation**

$2z' + \frac{x}{1-x^2}z = 0$ **is homogeneous equation**

$2\frac{dz}{dx} = -\frac{x}{1-x^2}z$ **is separated equation**

$$2\int \frac{dz}{z} = -\int \frac{xdx}{1 - x^2} \rightarrow \; 2lnz = \frac{1}{2}ln\left(c(1 - x^2)\right) \rightarrow$$

$$\rightarrow z = c\sqrt[4]{1 - x^2} = C(x)\sqrt[4]{1 - x^2},$$

substitute it into the initial nonhomogeneous equation $2z' + \frac{x}{1-x^2}z = x$**:**

$$2\left(C(x)\sqrt[4]{1 - x^2}\right)' + \frac{x}{1 - x^2}\left(C(x)\sqrt[4]{1 - x^2}\right) = x$$

$$2C'(x)\sqrt[4]{1 - x^2} + 2C(x)\frac{1}{4}(1 - x^2)^{\frac{1}{4}-1}(-2x) + \frac{x}{1 - x^2}\sqrt[4]{1 - x^2} \cdot C(x) = x$$

$$2C'(x)\sqrt[4]{1-x^2}+2C(x)\frac{-2x}{4\sqrt[4]{(1-x^2)^3}}+\frac{x}{1-x^2}\sqrt[4]{1-x^2}\cdot C(x)=x$$

divide by $\sqrt[4]{1-x^2}$

$$2C'(x)-C(x)\frac{x}{1-x^2}+\frac{x}{1-x^2}\cdot C(x)=\frac{x}{\sqrt[4]{1-x^2}}$$

$$2C'(x)=\frac{x}{\sqrt[4]{1-x^2}}$$

$$C(x)=-\frac{1}{4}\int(1-x^2)^{-\frac{1}{4}}(-2x)dx$$

$$C(x)=-\frac{1}{4}\frac{(1-x^2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}+c$$

$$C(x)=-\frac{1}{4}\frac{(1-x^2)^{\frac{3}{4}}}{\frac{3}{4}}+c$$

$$z=C(x)\sqrt[4]{1-x^2}=\left(-\frac{1}{4}\frac{(1-x^2)^{\frac{3}{4}}}{\frac{3}{4}}+c\right)\sqrt[4]{1-x^2}=-\frac{1-x^2}{3}+c\sqrt[4]{1-x^2}=$$

$$=\frac{x^2-1}{3}+c\sqrt[4]{1-x^2}$$

Recall the substitution $z=\sqrt{y}$ **and finally obtain that**

$$\sqrt{y}=c\sqrt[4]{1-x^2}-\frac{1}{3}(1-x^2)$$

or $y=c\sqrt{1-x^2}-\frac{2c}{3}(1-x^2)\sqrt[4]{1-x^2}+\frac{1}{9}(1-x^2)^2$

Method 3

$2z' + \frac{x}{1-x^2} z = x$ is a nonhomogeneous equation

$$z' + \frac{x}{2(1-x^2)} z = \frac{x}{2}$$

This is a linear first order ordinary differential equation in the dependent variable x. We write the equation in the form:

$$\frac{dz}{dx} + P(x) z = Q(x)$$

$$\frac{dz}{dx} + \frac{x}{2(1-x^2)} z = \frac{x}{2}$$

So $P(x) = \frac{x}{2(1-x^2)}$, $Q(x) = \frac{x}{2}$

The integrating factor is

$$\mu(x) = e^{\int P(x) dx}$$

that is,

$$\mu(x) = e^{\int \frac{x}{2(1-x^2)} dx}$$

$$\int \frac{x}{2(1-x^2)} dx = -\frac{1}{4} \int \frac{-2x}{1-x^2} dx = -\frac{1}{4} \ln(1-x^2)$$

$$\mu(x) = e^{-\frac{1}{4}\ln(1-x^2)} = \frac{1}{\sqrt[4]{1-x^2}}$$

Consider a differential equation

$$\frac{dz}{dx} + \frac{x}{2(1-x^2)} z = \frac{x}{2}$$

multiplying by $\frac{1}{\sqrt[4]{1-x^2}}$ **gives**

$$\frac{1}{\sqrt[4]{1-x^2}}\frac{dz}{dx}+\frac{\boldsymbol{x}}{\mathbf{2}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})\sqrt[4]{1-x^2}}\ z\ =\frac{\boldsymbol{x}}{\mathbf{2}\sqrt[4]{1-x^2}}$$

So , $\left(\frac{z}{\sqrt[4]{1-x^2}}\right)' = \frac{x}{2\sqrt[4]{1-x^2}}$

$$\left(\boldsymbol{z}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\mathbf{1}/\mathbf{4}}\right)' = \frac{\boldsymbol{x}}{\mathbf{2}\sqrt[4]{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}}}$$

integrating both sides with respect to x **gives**

$$\boldsymbol{z}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\mathbf{1}/\mathbf{4}} = \int \frac{\boldsymbol{x}\boldsymbol{d}\boldsymbol{x}}{\mathbf{2}\sqrt[4]{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}}}$$

$$\boldsymbol{z}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\mathbf{1}/\mathbf{4}} = -\frac{\mathbf{1}}{\mathbf{4}}\int (\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\frac{\mathbf{1}}{\mathbf{4}}}(-\mathbf{2}\boldsymbol{x})\boldsymbol{d}\boldsymbol{x}$$

$\boldsymbol{z}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\mathbf{1}/\mathbf{4}} = -\frac{1}{4}\frac{(1-x^2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + \boldsymbol{c},$

where c is an arbitrary real constant

$\boldsymbol{z}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{-\mathbf{1}/\mathbf{4}} = -\frac{1}{3}(\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{\frac{3}{4}} + \boldsymbol{c},$

whence

$\boldsymbol{z} = -\frac{1-x^2}{3} + \boldsymbol{c}\sqrt[4]{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}},$

$\boldsymbol{z} = \frac{x^2-1}{3} + \boldsymbol{c}\sqrt[4]{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}},$

where c **is an arbitrary real constant.**

The solution to a linear first order differential equation is then

$$\boldsymbol{z}(\boldsymbol{x}) = \frac{\int \boldsymbol{\mu}(\boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{d}\boldsymbol{x}+\boldsymbol{C}}{\boldsymbol{\mu}(\boldsymbol{x})}$$

$$\boldsymbol{z}(\boldsymbol{x}) = \frac{\int \frac{x}{2\sqrt[4]{1-x^2}}dx+C}{\frac{1}{\sqrt[4]{1-x^2}}} = \frac{-\frac{1}{3}(1-x^2)^{\frac{3}{4}}+c}{\frac{1}{\sqrt[4]{1-x^2}}} = \frac{x^2-1}{3} + \boldsymbol{c}\sqrt[4]{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}},$$

where c is an arbitrary real constant.

Recall the substitution $z = \sqrt{y}$ and finally obtain that

$$\sqrt{y} = c\sqrt[4]{1 - x^2} - \frac{1}{3}(1 - x^2),$$

where c is an arbitrary real constant

or $y = c\sqrt{1 - x^2} - \frac{2c}{3}(1 - x^2)\sqrt[4]{1 - x^2} + \frac{1}{9}(1 - x^2)^2$

Answer: $\sqrt{y} = c\sqrt[4]{1 - x^2} - \frac{1}{3}(1 - x^2), \quad y = 0.$