Answer on Question #52683 – Math – Linear Algebra

a) Show that, if A is any $n \times n$ matrix with real entries, then there is a $n \times n$ symmetric matrix S and a $n \times n$ skew symmetric matrix S' such that $A = S + S'$.

b) Find the solutions to the following system of equations by reducing the corresponding augmented matrix to row-reduced echelon form.

$2a + 3b + 4c + d = 8$

$a + 2b + 2c + 2d = 3$

$a - b + c + 3d = 3$

Solution

a) Let's rewrite A in the following way

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

By properties of transpose matrices, the following identities hold true:

$(A + B)^T = A^T + B^T$, $(A - B)^T = A^T - B^T$, $(A^T)^T = A$.

It's easy to verify that $S = \frac{A+A^T}{2}$ is symmetric and $S' = \frac{A-A^T}{2}$ is skew symmetric:

$$S^T = \left(\frac{A + A^T}{2}\right)^T = \frac{A^T + A}{2} = S$$

$$(S')^T = \left(\frac{A - A^T}{2}\right)^T = \frac{A^T - A}{2} = -\frac{A - A^T}{2} = -S'$$

b) The given system can be written in the matrix form as follows

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix}$$

The augmented matrix

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & -1 & 1 & 3 & 3 \end{array}\right)$$

Transformation to row echelon form:

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & -1 & 1 & 3 & 3 \end{array}\right) \xrightarrow{\text{subtract row 1 from doubled row 2 and from doubled row 3}} \left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & -5 & -2 & 5 & -2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & -5 & -2 & 5 & -2 \end{array}\right) \xrightarrow{\text{add 5 times row 2 to row 3}} \left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & -2 & 20 & -12 \end{array}\right) \xrightarrow{\text{divide row 3 by } (-2)}$$

$$\left(\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 8 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -10 & 6 \end{array}\right) \xrightarrow{\text{subtract 3 times row 2 and 4 times row 3 from row 1}}$$

$$\left(\begin{array}{cccc|c} 2 & 0 & 0 & 32 & -10 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -10 & 6 \end{array}\right) \xrightarrow{\text{divide row 1 by 2}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 16 & -5 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -10 & 6 \end{array}\right)$$

Therefore,

$$\begin{aligned} a + 16d &= -5 \\ b + 3d &= -2 \\ c - 10d &= 6 \end{aligned} \quad => \quad \begin{aligned} a &= -16d - 5 \\ b &= -3d - 2 \\ c &= 10d + 6 \end{aligned}$$

d is an arbitrary real constant.

Thus,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -16d - 5 \\ -3d - 2 \\ 10d + 6 \\ d \end{pmatrix} = \begin{pmatrix} -16 \\ -3 \\ 10 \\ 1 \end{pmatrix} d + \begin{pmatrix} -5 \\ -2 \\ 6 \\ 0 \end{pmatrix},$$

d is an arbitrary real constant.

Answer:

a) $S = \frac{A+A^T}{2}, S' = \frac{A-A^T}{2}$

b) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -16d - 5 \\ -3d - 2 \\ 10d + 6 \\ d \end{pmatrix} = \begin{pmatrix} -16 \\ -3 \\ 10 \\ 1 \end{pmatrix} d + \begin{pmatrix} -5 \\ -2 \\ 6 \\ 0 \end{pmatrix}$