Answer on Question #52681 – Math – Linear Algebra

a) Find the values of a, b ∈C for which the matrix

$$A = \begin{pmatrix} 1 & i & 1+i \\ a & 0 & b \\ 1-i & 2+i & 1 \end{pmatrix}$$

is Hermitian.

Solution

$$A^{\dagger} = \begin{pmatrix} 1 & a^* & 1+i \\ -i & 0 & 2-i \\ 1-i & b^* & 1 \end{pmatrix}.$$

If A is Hermitian

$$A = A^{\dagger}.$$

Thus

$$a = -i; b = 2 - i.$$

b) Are there values of a∈C for which the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & a \end{pmatrix}$$

is unitary? Justify your answer.

Solution

If A is unitary then

$$\det A = \pm 1.$$

$$\det A = 1\left(-\frac{1}{\sqrt{2}} \cdot a - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\left(a + \frac{1}{\sqrt{2}}\right).$$

If $\det A = -1$

$$\left(a + \frac{1}{\sqrt{2}}\right) = \sqrt{2} \ \rightarrow a = \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

If $\det A = 1$

$$\left(a + \frac{1}{\sqrt{2}}\right) = -\sqrt{2} \rightarrow a = -\sqrt{2} - \frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}.$$

A is unitary if $a = \pm\sqrt{2} - \frac{1}{\sqrt{2}}$.

But the matrix A is real and therefore it is orthogonal.

But orthogonal matrix should look like

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & \sin\varphi & -\cos\varphi \end{pmatrix}.$$

But in our case

$$-\sqrt{2} - \frac{1}{\sqrt{2}} \neq -\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

Thus, there is $a = \frac{1}{\sqrt{2}}$ ∈C for which the matrix A is unitary.

c) Let (x1,x2,x3) and (y1,y2,y3) represent the coordinates with respect to the bases B1={(1,0,0),(0,1,0),(0,0,1)}, B2={(1,0,0),(0,1,2),(0,2,1)}. If Q(X)=x21+2x1x2+2x2x3+x22+x23, find the representation of Q in terms of (y1,y2,y3).

Solution

We have a matrix Q

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We need to find the transition matrix.

$$\overline{b_{21}} = \overline{b_{11}}, \overline{b_{22}} = \overline{b_{12}} + 2\overline{b_{13}}, \overline{b_{23}} = 2\overline{b_{12}} + \overline{b_{13}}.$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Inverting, $\overline{b_{11}} = \overline{b_{21}}, \overline{b_{12}} = -\frac{1}{3}\overline{b_{22}} + \frac{2}{3}\overline{b_{23}}, \overline{b_{23}} = \frac{2}{3}\overline{b_{22}} - \frac{1}{3}\overline{b_{23}}.$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}.$$

Thus

$$Q_{B_2} = P^{-1}QP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -\frac{1}{3} & 1 & 1 \\ \frac{2}{3} & 1 & 1 \end{pmatrix}.$$

Thus the representation of Q in terms of (y1,y2,y3) is

$$Q' = y_1^2 + y_2^2 + y_3^2 + \frac{2}{3}y_1y_2 + \frac{8}{3}y_1y_3 + 2y_2y_3.$$