Answer on Question #52680 – Math – Linear Algebra

a) For the following matrices, check whether there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. When such a P exists, find P.

(i) $A = \begin{pmatrix} 0 & 1 & -3 \\ 2 & -1 & 6 \\ 1 & -1 & 4 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

b) Find the inverse of the matrix B in part (a) by using Cayley-Hamilton theorem.

c) Using the fact that $\det(AB) = \det(A)\det(B)$ for any two matrices A and B, prove the identity $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$

Solution

a) Such matrix P exists if there are no zero vectors among eigenvectors of given matrix (A or B)

(i) The columns of matrix P are given by the eigenvectors of the matrix A. Let's first find its eigenvalues. The characteristic polynomial of A:

$$p(\lambda) = \det(\lambda I_3 - A) = \det\begin{pmatrix} \lambda & -1 & 3 \\ -2 & \lambda + 1 & -6 \\ -1 & 1 & \lambda - 4 \end{pmatrix} = \lambda^3 - 3\lambda^2 + 7\lambda - 5$$

Since eigenvalues satisfy the equation $p(\lambda) = 0$, we obtain the following eigenvalues

$$\lambda_{1,2,3} = 1$$

The eigenvectors are

$\lambda = 1$:

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore there is no such matrix for A.

(ii) The columns of matrix P are given by the eigenvectors of the matrix B. Let's first find its eigenvalues. The characteristic polynomial of B:

$$p(\lambda) = \det(\lambda I_3 - B) = \det\begin{pmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{pmatrix} = \lambda^3 - 3\lambda^2 + 7\lambda - 5$$

Since eigenvalues satisfy the equation $p(\lambda) = 0$, we obtain the following eigenvalues

$$\lambda_{1,2,3} = 1, 2, 2$$

The eigenvectors are

$\lambda = 1$:

$$e_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$:

$$e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

b) The characteristic polynomial of B is given by

$$p(\lambda) = \det(\lambda I_3 - B)$$

The Cayley-Hamilton theorem states that

$$p(B) = 0$$

Since

$$p(\lambda) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4,$$

we obtain

$$p(B) = B^3 - 5B^2 + 8B - 4I_3 = 0$$

Multiplying by B^{-1} we obtain the following

$$B^2 - 5B + 8I_3 - 4B^{-1} = 0$$

Therefore

$$B^{-1} = \frac{1}{4}(B^2 - 5B + 8I_3)$$

Since

$$B^2 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix},$$

we obtain

$$B^{-1} = \frac{1}{4}\left(\begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix} - 5\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) =$$

$$= \frac{1}{4}\begin{pmatrix} 6 & 0 & 4 \\ -2 & 2 & -2 \\ -2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/2 & 0 & 1 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 \end{pmatrix}$$

c) Let's consider two matrices:

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \qquad B = \begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} a & b \\ d & c \end{pmatrix}\begin{pmatrix} a & d \\ b & c \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ad + bc \\ ad + bc & c^2 + d^2 \end{pmatrix}$$

Since

$$\det(AB) = \det(A)\det(B),$$

we obtain

$$\det\begin{pmatrix} a^2 + b^2 & ad + bc \\ ad + bc & c^2 + d^2 \end{pmatrix} = \det\begin{pmatrix} a & b \\ d & c \end{pmatrix}\det\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

$$(a^2 + b^2)(c^2 + d^2) - (ad + bc)^2 = (ac - bd)(ac - bd)$$

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Answer:

a)

(i) Such a matrix does not exist

(ii) $\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 3/2 & 0 & 1 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 \end{pmatrix}$