Answer on Question #52674 – Math – Multivariable Calculus

i) Determine which of the vector fields given below is conservative

a) $F = (\frac{y}{x^2+y^2} - 1)i - \frac{x}{x^2+y^2}j$

b) $F = (\frac{x}{x^2+y^2} - 1)i - \frac{y}{x^2+y^2}j$

ii) Show the points $A\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$ and $B(-\pi, \pi)$ on the x,y-plane

iii) Calculate the work done (in dimensionless units) by the conservative field between these points, while presenting answer in exact form and then calculate up to four significant figures

Solution:

i) For the field to be conservative the following condition should be met

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

a) Let's check if this condition holds for F

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2} - 1\right) = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = -\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Since $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$, the given vector field is conservative.

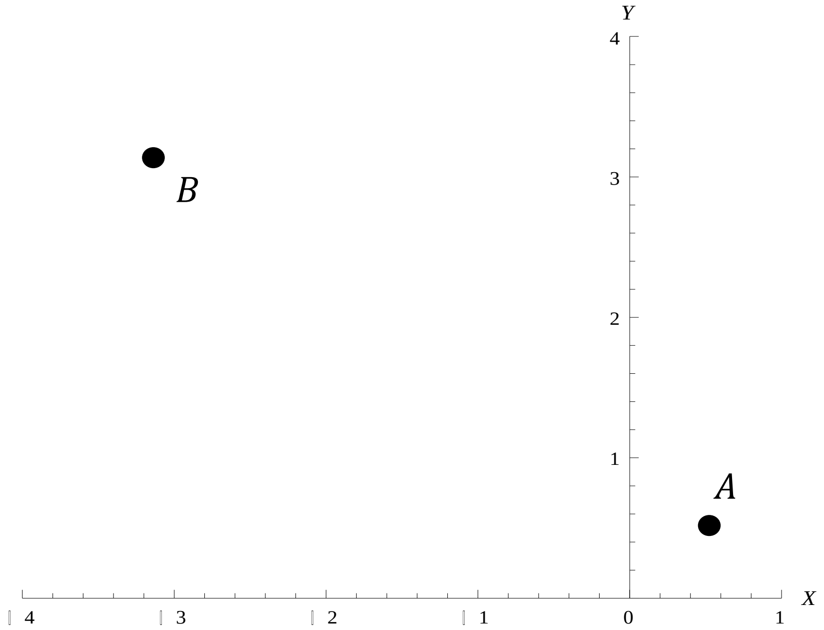
b) Let's check if this condition holds for F

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2} - 1\right) = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

Since $\frac{\partial F_x}{\partial y} \neq \frac{\partial F_y}{\partial x}$, the given vector field is not conservative.

ii)



iii) It's easy to verify that this field has the following potential function

$$V = x - \arctan\left(\frac{x}{y}\right)$$

Indeed

$$F_x = -\frac{\partial V}{\partial x} = -1 + \frac{\frac{1}{y}}{1 + \frac{x^2}{y^2}} = \frac{y}{x^2 + y^2} - 1$$

$$F_y = -\frac{\partial V}{\partial y} = \frac{-\frac{x}{y^2}}{1 + \frac{x^2}{y^2}} = -\frac{x}{x^2 + y^2}$$

Therefore, the work done by field between points A and B is given by

$$W = V(B) - V(A) = -\pi - \arctan\left(\frac{-\pi}{\pi}\right) - \left(\frac{\pi}{6} - \arctan\left(\frac{\pi/6}{\pi/6}\right)\right) =$$

$$= -\pi - \arctan(-1) - \left(\frac{\pi}{6} - \arctan(1)\right) = -\pi - \left(-\frac{\pi}{4}\right) - \left(\frac{\pi}{6} - \frac{\pi}{4}\right) = -\frac{2}{3}\pi$$

Answer:

i) a)

iii) $-\frac{2}{3}\pi$