Answer on Question #52638 – Math – Combinatorics | Number Theory

If $(a,b)=1$ then prove that $(a-b,a+b,ab)=1$

Solution

Let a,b be natural numbers.
Let's prove it by contradiction. Suppose that there exists natural c such that $(a-b,a+b,ab)=c$. This means that there exist natural m,k and n such that $a-b=nc$ and $a+b=mc$ and $ab=kc$.
Thus, $(a-b)+(a+b)=(nc)+(mc)\Rightarrow 2a=c(n+m)\Rightarrow n+m=\frac{2a}{c}$. Since m and n are natural, then $(n+m)$ is natural. This means that 1) $c=2$ or 2) c has a divisor d such that $d\,|\,a$.
In the same time we have $(a+b)-(a-b)=(mc)-(nc)\Rightarrow 2b=c(m-n)\Rightarrow m-n=\frac{2b}{c}$. Since m and n are natural, then $(m-n)$ is natural. This means that 1) $c=2$ or 2) c has a divisor d such that $d\,|\,b$.
Due to conditions of the question, we have $(a,b)=1$, therefore $c=2$. So, $ab=2k$, i.e. ab is even, this means that some of numbers a, b is even. Assume that b is even, i.e. there exists natural l such that $b=2l$.
Since $a-b=nc\Rightarrow a-2l=2n\Rightarrow a=2(n+l)$, hence a is even.
Thus, we obtain that a and b are even numbers. So we come to a contradiction with $(a,b)=1$. So initial assumption $(a-b,a+b,ab)=c$ does not hold true.
This means that if $(a,b)=1$ then $(a-b,a+b,ab)=1$.