Answer on Question #52635 – Math – Multivariable Calculus

1) Use spherical coordinates to calculate the triple integral of $f(x, y, z)$ over the given region D

a. $f(x,y,z) = y,\ D: x^2 + y^2 + z^2 \leq 1,\ x, y, z \leq 0$

b. $f(x,y,z) = (x^2 + y^2 + z^2)^{-\frac{3}{2}},\ D: 2 \leq x^2 + y^2 + z^2 \leq 4$

2) Find a potential function for $(yz^2, xz^2, 2xyz)$

Solution:

1) The spherical coordinates

$$x = r \sin\phi \cos\theta$$
$$y = r \sin\phi \sin\theta,$$
$$z = r\cos\phi$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\phi \in [0, \pi]$.

The Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = r^2 \sin\phi.$$

a. Let's express D in terms of spherical coordinates:

$$D: r \leq 1, \theta \in \left[\pi, \frac{3\pi}{2}\right], \phi \in \left[\frac{\pi}{2}, \pi\right].$$

f in spherical coordinates is given by

$$f(r,\theta,\phi) = r\sin\phi\sin\theta$$

The triple integral of f over D is given by

$$\iiint_D f(r,\theta,\phi) r^2 \sin\phi \, dr d\theta d\phi =$$

$$= \iiint_D r^3 \sin^2\phi \sin\theta \, dr d\theta d\phi = \left(\int_0^1 r^3 dr\right)\left(\int_\pi^{\frac{3\pi}{2}} \sin\theta \, d\theta\right)\left(\int_{\frac{\pi}{2}}^{\pi} \sin^2\phi \, d\phi\right) =$$

$$= \left(\frac{r^4}{4}\Big|_0^1\right)\left(-\cos\theta\Big|_\pi^{\frac{3\pi}{2}}\right)\left(\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\phi}{2} d\phi\right) = \frac{1}{4}\left(\frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\frac{\pi}{2}}^{\pi}\right) =$$

$$= \frac{1}{4}\left(\frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\frac{\pi}{2}}^{\pi}\right) = \frac{\pi}{16}.$$

b. Let's express D in terms of spherical coordinates:

$$D: \sqrt{2} \le r \le 2, \theta \in [0, 2\pi), \phi \in [0, \pi].$$

f in spherical coordinates is given by

$$f(r, \theta, \phi) = \frac{1}{r^3}$$

The triple integral of f over D is given by

$$\iiint_D f(r, \theta, \phi) r^2 \sin\phi \, dr d\theta d\phi =$$

$$= \iiint_D \frac{1}{r} \sin\phi \, dr d\theta d\phi = \left(\int_{\sqrt{2}}^{2} \frac{1}{r} dr\right)\left(\int_0^{2\pi} 1 d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right) =$$

$$= \left(\ln r|_{\sqrt{2}}^{2}\right)\left(\theta|_0^{2\pi}\right)\left(-\cos\phi|_0^{\pi}\right) = \ln\left(\sqrt{2}\right) 2\pi \cdot 2 = 2\pi \ln 2.$$

2) We'll assume that $f = \nabla V$, then

$$yz^2 = \frac{\partial V}{\partial x}$$

$$xz^2 = \frac{\partial V}{\partial y}$$

$$2xyz = \frac{\partial V}{\partial z}$$

Integrating the first of these equations we obtain

$$V = xyz^2 + C(y, z)$$

Substituting this into the 2nd and 3rd equations we obtain

$$xz^2 = xz^2 + \frac{\partial C}{\partial y}$$

$$2xyz = 2xyz + \frac{\partial C}{\partial z}$$

that is,

$$\frac{\partial C}{\partial y} = 0$$

$$\frac{\partial C}{\partial z} = 0$$

$C = 0$ satisfies both of these equations. Therefore,

$$V = xyz^2$$

Answer:

1)

a. $\frac{\pi}{16}$

b. $2\pi \ln 2$

2) $V = xyz^2$