Answer on Question #52633 - Math – Multivariable Calculus

1) Find the total mass of the square disk $0 < x, y < 1$ if the mass density is $f(x, y) = x^2 + y^2$

2) Calculate the average value of the function $f\ (x, y) = e^{x+y}$ on the square $[0,1] \times [0,1]$

3) Calculate the average height above the x-axis of a point in the region $0 \le x \le 1, 0 \le y \le x^2$

Solution:

1) The mass of the square disk is given by

$$m = \iint\limits_{0<x,y<1} f(x,y)dxdy = \int_0^1 \left(\int_0^1 (x^2 + y^2)dx \right) dy =$$

$$= \int_0^1 \left(y^2 x + \frac{x^3}{3} \right) \Bigg|_{x=0}^{x=1} dy = \int_0^1 \left(y^2 \cdot 1 + \frac{1}{3} - y^2 \cdot 0 - \frac{0^3}{3} \right) dy = \int_0^1 \left(y^2 + \frac{1}{3} \right) dy =$$

$$= \left(\frac{y^3}{3} + \frac{1}{3} y \right) \Bigg|_{y=0}^{y=1} = \frac{1^3}{3} + \frac{1}{3} \cdot 1 - \frac{0^3}{3} - \frac{1}{3} \cdot 0 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

2) The average is given by

$$\overline{f} = \frac{\iint_{0\,0}^{1\,1} f(x,y)dxdy}{\iint_{0\,0}^{1\,1} dxdy} = \frac{\left(\int_0^1 e^x dx\right)\left(\int_0^1 e^y dy\right)}{(1-0)\cdot(1-0)} = \left(\int_0^1 e^x dx \right)^2 = (e^x|_0^1)^2 = (e^1 - e^0)^2$$

$$= (e-1)^2$$

3) The area of the region is

$$S = \int_0^1 x^2 dx = \frac{x^3}{3}\Bigg|_0^1 = \frac{1}{3}$$

The average height is

$$h = \frac{\iint y dxdy}{S},$$

where the double integral is taken over the given region. The previous equation can be rewritten in the following way:

$$h = \frac{\iint y dxdy}{S} = \frac{\int_0^1 \left(\int_0^{x^2} y dy \right) dx}{S} = \frac{\int_0^1 \left(\frac{1}{2} y^2 \right) \Big|_0^{x^2} dx}{S} =$$

$$= \frac{\int_0^1 \frac{1}{2} x^4 dx}{S} = \frac{\frac{1}{2}\frac{x^5}{5}\Big|_0^1}{S} = \frac{1/10}{1/3} = \frac{3}{10}.$$

Answer:

1) $\frac{2}{3}$
2) $(e-1)^2$
3) $\frac{3}{10}$