Answer on Question #52628 - Math - Vector Calculus

1) Verify that the vector fields are conservation by comparing cross derivatives, then potential functions for the, by taking anti-derivatives

A) $f = (z, 1, x)$

B) $f = (y^2, 2xy + e^z, ye^z)$

C) $f = (\cos z\,, 2y, -x \sin z)$

Solution

Curl is given by

$$\nabla \times f = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left[\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right]\vec{\imath} + \left[\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right]\vec{\jmath} + \left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right]\vec{k}$$

A) Since

$$\nabla \times f = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 1 & x \end{vmatrix} = \left[\frac{\partial x}{\partial y} - \frac{\partial 1}{\partial z}\right]\vec{\imath} + \left[\frac{\partial z}{\partial z} - \frac{\partial x}{\partial x}\right]\vec{\jmath} + \left[\frac{\partial 1}{\partial x} - \frac{\partial z}{\partial y}\right]\vec{k} =$$

$$= [0-0]\vec{\imath} + [1-1]\vec{\jmath} + [0-0]\vec{k} = \vec{0},$$

the vector field f is conservative. It's easy to verify ($f = \nabla V$) that its potential function is given by

$V = xz + y + C$ (where C is an arbitrary real constant), because

$\nabla V = \left(\frac{\partial(xz+y+C)}{\partial x}; \frac{\partial(xz+y+C)}{\partial y}; \frac{\partial(xz+y+C)}{\partial z}\right) = (z, 1, x).$

To find function V, solve the following system

$\frac{\partial V}{\partial x} = f_x, \frac{\partial V}{\partial y} = f_y, \frac{\partial V}{\partial z} = f_z$; that is,

$\frac{\partial V}{\partial x} = z, \frac{\partial V}{\partial y} = 1, \frac{\partial V}{\partial z} = x;$

For $\frac{\partial V}{\partial x} = z$ integrate both sides with respect to x and obtain $V = zx + g(y,z)$. Taking the partial derivative of the both sides with respect to y obtain

$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(zx + g(y,z)) = \frac{\partial g(y,z)}{\partial y}.$

On the other hand, taking into account the system, $\frac{\partial V}{\partial y} = 1.$

Equating right-hand sides of two formulas gives $\frac{\partial g(y,z)}{\partial y} = 1$.

Integrating both sides with respect to y obtain $g(y,z) = y + C(z)$, hence

$V = zx + g(y,z) = zx + y + C(z)$.

Taking the partial derivative of the both sides with respect to z obtain

$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(zx + y + C(z)) = x + C'(z)$.

On the other hand, taking into account the system, $\frac{\partial V}{\partial z} = x$.

Equating right-hand sides of two formulas gives $x + C'(z) = x$, hence $C'(z) = 0$, integrating with respect to z gives $C(z) = C$, where C is an arbitrary real constant.

Thus, $V = zx + g(y,z) = zx + y + C(z) = zx + y + C$.

B) Since

$$\nabla \times f = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + e^z & ye^z \end{vmatrix} = \left[\frac{\partial(ye^z)}{\partial y} - \frac{\partial(2xy + e^z)}{\partial z}\right]\vec{\imath} + \left[\frac{\partial(y^2)}{\partial z} - \frac{\partial(ye^z)}{\partial x}\right]\vec{\jmath} + \left[\frac{\partial(2xy + e^z)}{\partial x} - \frac{\partial(y^2)}{\partial y}\right]\vec{k}$$

$$= [e^z - e^z]\vec{\imath} + [0 - 0]\vec{\jmath} + [2y - 2y]\vec{k} = \vec{0}$$

the vector field f is conservative. It's easy to verify ($f = \nabla V$) that its potential function is given by

$V = xy^2 + ye^z + C$ (where C is an arbitrary real constant),

because

$\nabla V = \left(\frac{\partial(xy^2+ye^z+C)}{\partial x}; \frac{\partial(xy^2+ye^z+C)}{\partial y}; \frac{\partial(xy^2+ye^z+C)}{\partial z}\right) = (y^2, e^z, ye^z)$.

To find function V, solve the following system

$\frac{\partial V}{\partial x} = f_x, \frac{\partial V}{\partial y} = f_y, \frac{\partial V}{\partial z} = f_z$; that is,

$\frac{\partial V}{\partial x} = y^2, \frac{\partial V}{\partial y} = 2xy + e^z, \frac{\partial V}{\partial z} = ye^z$;

For $\frac{\partial V}{\partial x} = y^2$ integrate both sides with respect to x and obtain $V = xy^2 + g(y,z)$. Taking the partial derivative of the both sides with respect to y obtain

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(xy^2 + g(y,z)) = 2xy + \frac{\partial g(y,z)}{\partial y}.$$

On the other hand, taking into account the system, $\frac{\partial V}{\partial y} = 2xy + e^z$.

Equating right-hand sides of two formulas gives $\frac{\partial g(y,z)}{\partial y} = e^z$.

Integrating both sides with respect to y obtain $g(y,z) = ye^z + C(z)$, hence

$$V = xy^2 + g(y,z) = xy^2 + ye^z + C(z).$$

Taking the partial derivative of the both sides with respect to z obtain

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(xy^2 + ye^z + C(z)) = ye^z + C^{'}(z).$$

On the other hand, taking into account the system, $\frac{\partial V}{\partial z} = ye^z$.

Equating right-hand sides of two formulas gives $ye^z + C^{'}(z) = ye^z$, hence $C^{'}(z) = 0$, integrating with respect to z gives $C(z) = C$, where C is an arbitrary real constant.

Thus, $V = xy^2 + g(y,z) = xy^2 + ye^z + C(z) = xy^2 + ye^z + C$

C) Since

$$\nabla \times f = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos z & 2y & -x\sin z \end{vmatrix} =$$

$$= \left[\frac{\partial(-x\sin z)}{\partial y} - \frac{\partial(2y)}{\partial z}\right]\vec{\imath} + \left[\frac{\partial(\cos z)}{\partial z} - \frac{\partial(-x\sin z)}{\partial x}\right]\vec{\jmath} + \left[\frac{\partial(2y)}{\partial x} - \frac{\partial(\cos z)}{\partial y}\right]\vec{k}$$

$$= [0-0]\vec{\imath} + [-\sin z - (-\sin z)]\vec{\jmath} + [0-0]\vec{k} = \vec{0},$$

the vector field f is conservative. It's easy to verify ($f = \nabla V$) that its potential function is given by

$V = y^2 + x\cos z + C$ (where C is an arbitrary real constant), because

$$\nabla V = \left(\frac{\partial(y^2 + x\cos z + C)}{\partial x}; \frac{\partial(y^2 + x\cos z + C)}{\partial y}; \frac{\partial(y^2 + x\cos z + C)}{\partial z}\right) = (\cos z, 2y, -x\sin z).$$

To find function V, solve the following system

$\frac{\partial V}{\partial x} = f_x, \frac{\partial V}{\partial y} = f_y, \frac{\partial V}{\partial z} = f_z$; that is,

$\frac{\partial V}{\partial x} = \cos z, \frac{\partial V}{\partial y} = 2y, \frac{\partial V}{\partial z} = -x\sin z;$

For $\frac{\partial V}{\partial x} = \cos z$ integrate both sides with respect to x and obtain $V = x\cos z + g(y,z)$. Taking the partial derivative of the both sides with respect to y obtain

$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(x\cos z + g(y,z)) = \frac{\partial g(y,z)}{\partial y}.$

On the other hand, taking into account the system, $\frac{\partial V}{\partial y} = 2y.$

Equating right-hand sides of two formulas gives $\frac{\partial g(y,z)}{\partial y} = 2y.$

Integrating both sides with respect to y obtain $g(y,z) = y^2 + C(z)$, hence

$V = x\cos z + g(y,z) = x\cos z + y^2 + C(z).$

Taking the partial derivative of the both sides with respect to z obtain

$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(x\cos z + y^2 + C(z)) = -x\sin z + C^{'}(z).$

On the other hand, taking into account the system, $\frac{\partial V}{\partial z} = -x\sin z$.

Equating right-hand sides of two formulas gives $-x\sin z + C^{'}(z) = -x\sin z$, hence $C^{'}(z) = 0$, integrating with respect to z gives $C(z) = C$, where C is an arbitrary real constant.

Thus, $V = x\cos z + g(y,z) = x\cos z + y^2 + C(z) = x\cos z + y^2 + C$

Answer:

A) $xz + y + C$

B) $xy^2 + ye^z + C$

C) $x\cos z + y^2 + C$