Answer on Question #52576 – Math – Abstract Algebra

Question. Let M and N be two subgroups of a soluble group G. Then show that MN is also soluble.

Solution. Recall that a group G is called *soluble* if there exists an increasing finite sequence of subgroups of G

$$\{1\} = G_0 < G_1 < \cdots < G_{n-1} < G_n = G,$$

such that G_i is normal in G_{i+1} for $i = 0, \ldots, n-1$, and the factor groups G_{i+1}/G_i is abelian. This sequence is called *subnormal series*.

We claim that *every subgroup H of a soluble group G is soluble as well.* In particular, this will imply that so is MN.

Let

$$\{1\} = G_0 < G_1 < \cdots < G_{n-1} < G_n = G,$$

be a subnormal series for G. Denote

$$H_i = H \cap G_i.$$

Then

$$H_0 = H \cap G_0 = H \cap \{1\} = \{1\},$$

and

$$H_n = H \cap G_n = H \cap G = H.$$

We will show that then the sequence

$$\{1\} = H_0 < H_1 < \cdots < H_{n-1} < H_n = H,$$

is a the subnormal series for H. For this we should check the following two statements.

1) *H_i is normal subgroup of H_{i+1}, that is $hH_ih^{-1} = H_i$ for all $h \in H_{i+1}$.*

Indeed, by assumption G_i is a normal subgroup of G_{i+1}, so $hG_ih^{-1} = G_i$ for all $h \in G_{i+1}$. Hence if $h \in H_{i+1} = H \cap G_{i+1}$, then

$$hH_ih^{-1} = h(H \cap G_i)h^{-1} = hHh^{-1} \cap hG_ih^{-1} = H \cap G_i = H_i.$$

2) *H_{i+1}/H_i is an abelan group.*

Indeed, by assumption the quotient group G_{i+1}/G_i is abelian. We will show that H_{i+1}/H_i can be identified with a subgroup of G_{i+1}/G_i which will imply that H_{i+1}/H_i is abelian as well.

Let $j : H_{i+1} = H \cap G_{i+1} \subset G_{i+1}$ be the inclusion map and $p : G_{i+1} \to G_{i+1}/G_i$ be the quotient map. Consider the composition

$$p \circ j : H_{i+1} \xrightarrow{j} G_{i+1} \xrightarrow{p} G_{i+1}/G_i.$$

Then $\ker(p \circ j) = H \cap G_i = H_i$. Hence $p \circ j$ induces an isomorphism

$$H_{i+1}/\ker(p \circ j) = H_{i+1}/H_i \;\cong\; p \circ j(H_{i+1}).$$

In other words, H_{i+1}/H_i is isomorphic with a subgroup of an abelian group G_{i+1}/G_i, whence H_{i+1}/H_i is abelian as well.

Thus $\{1\} = H_0 < H_1 < \cdots < H_{n-1} < H_n = H$ is a subnormal series for H, and so H is soluble.

In particular, if M and N are two subgroups of a soluble group G, then M, N and MN are soluble as well.