Answer on Question #52572 – Math - Combinatorics | Number Theory

Question

Show that if (b,c) =1 then (a,bc)=(a,b)(a,c).

Solution

Method 1.

If $(a,b) = xa + yb$, $(a,c) = sa + tc$, then

$$(a,b)(a,c) = (xa + yb)(sa + tc) = xsaa + xtac + ysab + ytbc =$$

$$= (xsa + xtc + ysb)a + (yt)bc.$$

By Bezout theorem, for nonzero a, bc there exist integers p, r such that

$(a,bc) = pa + r(bc)$.

Denote integers by $p = xsa + xtc + ysb$ and $r = yt$.

Thus, (a,bc)=(a,b)(a,c) .

Method 2.

If $(m,n) = z$, then $z|m$, $z|n$ and it does not hold true that simultaneously $(kz)|m$, $(kz)|n$, k is integer, $k \neq 0, k \neq 1, k \neq -1$. It means that $m = zm_1$, $n = zn_1$, where m_1 and n_1 do not have common divisors. Multiply both equalities by a and obtain $am = azm_1$, $an = azn_1$, besides, $(am, an) = az$, because m_1 and n_1 do not have common divisors.

Thus, we know the property:

$$(am,an) = a(m,n)$$

Next,

$$(a,b) = x \Rightarrow a = xa_1, b = xb_1$$
$$(a,c) = y \Rightarrow a = ya_2, c = yc_1$$
$$ya_2 = xa_1 \Rightarrow xa_1 \vdots y \Rightarrow a_1 \vdots y, \ because (b,c) = (xb_1, yc_1) = 1 \Rightarrow a_1 = a_3 y$$
$$(a,bc) = (xa_1, xb_1 yc_1) = x(a_1, yb_1c_1) = x(a_3 y, yb_1c_1) = xy(a_3, b_1c_1) = xy = (a,b)(a,c)$$