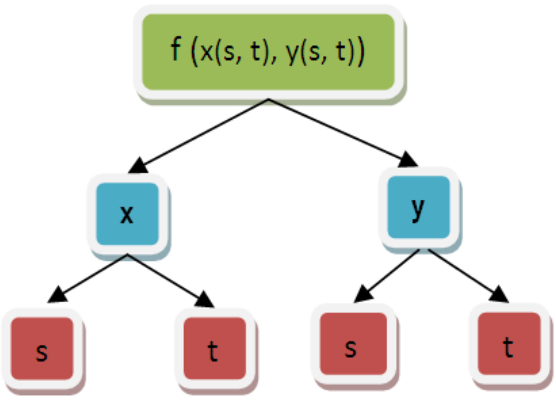
Answer on Question #52434 – Math – Multivariable Calculus

Let $x = s + t$, and $y = s - t$. Show that $\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial f}{\partial t}\right)$.

Solution

We have $f = f(x, y), x = x(s,t), y = y(s,t)$. So, in this case, f is a function of s and t. There are two partial derivatives to compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Therefore, we have to use the next tree structures to compute partial derivatives:



The first partials of f with respect to s or t are computed as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \left| \frac{\partial x}{\partial s} = 1, \frac{\partial y}{\partial s} = 1 \right| = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left| \frac{\partial x}{\partial t} = 1, \frac{\partial y}{\partial t} = -1 \right| = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.$$

Then

$$\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\cdot\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial x}\right)^2 \underbrace{- \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial x}}_{0} - \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2.$$