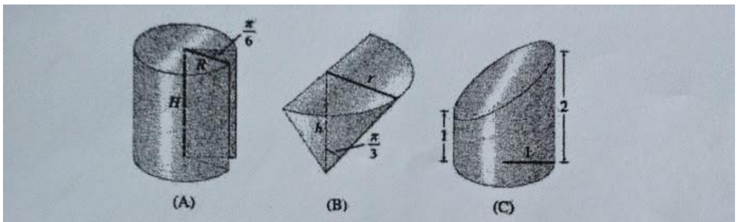
Answer on Question #52229 - Math - Multivariable Calculus

1)



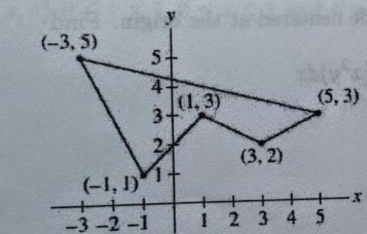
(A) (B) (C)

Let Oz be the vertical line thru the center of the circular base, the Oxy plane is the bottom of the solid, and the Oxz plane divide the solids into symmetric halves.

(Hint: Calculate $V = \int_W dw$, $V_x = \int_W x\, dw$, $V_y = \int_W y\, dw$, $V_z = \int_W z\, dw$ and the centroid is at $(\frac{V_x}{V}, \frac{V_y}{V}, \frac{V_z}{V})$)

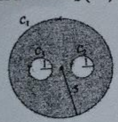
A.

2)



b. Compute the area of the following polygon by the integral of $\frac{1}{2}(-y, x)$ on the boundary.

3)



69. Evaluate the circulation of F around C_1, $\oint_{C_1} F \cdot ds$, if

$$\oint_{C_2} F \cdot ds = 3\pi \qquad \oint_{C_3} F \cdot ds = 4\pi$$

and $\mathbf{curl}_z(F) \equiv 9$ on the shaded region

Solution:

1)

(A) Since the cross-section of the solid doesn't change along the z-axis, z-coordinate of the centroid is $H/2$ (half-height). Since Oxz plane divides the solid into symmetric halves, the y-coordinate of the centroid is 0. Therefore, we should only calculate V_x and V. The cross-section of the base is given by

$$A_b = \frac{11}{12}\pi R^2$$

Thus, the volume of the solid is

$$V = H \cdot A_b = \frac{11}{12}\pi H \cdot R^2$$

Let's now calculate V_x:

$$V_x = \int_W x dw = \left| \begin{matrix} dw = 2H\sqrt{R^2 - x^2}dx & x \in [-R, 0] \\ dw = 2H\left[\sqrt{R^2 - x^2} - x\tan\frac{\pi}{12}\right] & x \in \left[0, R\cos\frac{\pi}{12}\right] \end{matrix} \right| =$$

$$= \int_{-R}^{0} x2H\sqrt{R^2 - x^2}dx + \int_{0}^{R\cos\frac{\pi}{12}} x2H\left[\sqrt{R^2 - x^2} - x\tan\frac{\pi}{12}\right]dx =$$

$$= 2H\int_{-R}^{R\cos\frac{\pi}{12}} x\sqrt{R^2 - x^2}dx - 2H\int_{0}^{R\cos\frac{\pi}{12}} x^2\tan\frac{\pi}{12}dx =$$

$$= H\int_{-R}^{R\cos\frac{\pi}{12}} \sqrt{R^2 - x^2}d(x^2) - 2H\tan\frac{\pi}{12}\int_{0}^{R\cos\frac{\pi}{12}} x^2 dx =$$

$$= -\frac{2}{3}H\left(R^2 - x^2\right)^{\frac{3}{2}}\Big|_{-R}^{R\cos\frac{\pi}{12}} - \frac{2}{3}H\tan\frac{\pi}{12}x^3\Big|_0^{R\cos\frac{\pi}{12}} =$$

$$= -\frac{2}{3}HR^3\sin^3\frac{\pi}{12} - \frac{2}{3}HR^3\tan\frac{\pi}{12}\cos^3\frac{\pi}{12} = -\frac{2}{3}HR^3\sin\frac{\pi}{12}$$

Therefore, the x-coordinate of the centroid is

$$\frac{V_x}{V} = \frac{-\frac{2}{3}HR^3\sin\frac{\pi}{12}}{\frac{11}{12}\pi H \cdot R^2} = -\frac{8}{11\pi}R\sin\frac{\pi}{12}$$

(B) Since Oxz plane divides the solid into symmetric halves, the y-coordinate of the centroid is 0. Therefore, we should only calculate V_x and V. The cross-section of the base is given by

$$A_b = \frac{\pi}{2}r^2$$

Thus, the volume of the solid is

$$V = \frac{1}{3}h \cdot A_b = \frac{\pi}{6}h \cdot r^2$$

Since $\frac{r}{h} = \tan\frac{\pi}{3} = \sqrt{3}$, we obtain

$$V = \frac{\pi}{2}h^3$$

Let's first calculate V_z:

$$V_z = \int_{W} z\,dw = \left| dw = \frac{\pi}{2}\tan^2\frac{\pi}{3} z^2 dz \qquad z \in [0, h] \right| =$$

$$= \int_0^h \frac{\pi}{2}\tan^2\frac{\pi}{3} z^3 dz = \frac{\pi}{2}\tan^2\frac{\pi}{3}\int_0^h z^3 dz = \frac{3\pi}{8}h^4$$

Therefore, the x-coordinate of the centroid is

$$\frac{V_z}{V} = \frac{\frac{3\pi}{8}h^4}{\frac{\pi}{2}h^3} = \frac{3}{4}h$$

Let's now calculate V_x. Since the centroid of the semicircle is situated on the line of symmetry $\frac{4}{3\pi}$ of the radius from its center, V_x is given by

$$V_x = \int_0^h \left(\frac{4}{3\pi} z \tan\frac{\pi}{3}\right)\frac{\pi}{2}\tan^2\frac{\pi}{3} z^2 dz =$$

$$= \frac{2}{3}\tan^3\frac{\pi}{3}\int_0^h z^3 dz = \frac{\sqrt{3}}{2}h^4$$

Therefore, the x-coordinate of the centroid is

$$\frac{V_x}{V} = \frac{\frac{\sqrt{3}}{2}h^4}{\frac{\pi}{2}h^3} = \frac{\sqrt{3}}{\pi}h$$

(C) Since Oxz plane divides the solid into symmetric halves, the y-coordinate of the centroid is 0. Let the full height of the solid be denoted as h ($h = 2$) and the radius of the base be denoted as r ($r = 1$). The area of the base is

$$A_b = \pi r^2 = \pi$$

The volume of the solid is

$$V = \frac{h}{2}A_b + \frac{1}{2}\left(\frac{h}{2}A_b\right) = \frac{3}{4}h \cdot A_b = \frac{3}{2}\pi$$

Let's first calculate V_x:

$$V_x = \int_{-r}^{r} x 2\sqrt{r^2 - x^2}\left(\frac{3}{4}h + \frac{1}{2}x\right)dx =$$

$$= \frac{3}{2}\int_{-1}^{1}\sqrt{r^2 - x^2}\,d(x^2) + \int_{-1}^{1} x^2\sqrt{r^2 - x^2}\,dx = \frac{\pi}{8}$$

Therefore, the z-coordinate of the centroid is

$$\frac{V_x}{V} = \frac{\frac{\pi}{8}}{\frac{3}{2}\pi} = \frac{1}{12}$$

Let's now calculate V_z:

$$V_z = \int_0^{\frac{h}{2}} z\pi r^2 dz + \int_{\frac{h}{2}}^{h} z\left\{\frac{\pi}{2} - \arctan\frac{\sqrt{1-(2z-3)^2}}{2z-3} - (2z-3)\sqrt{1-(2z-3)^2}\right\}dz =$$

$$=\frac{\pi}{2}+\frac{21}{32}\pi=\frac{37}{32}\pi$$

Therefore, the z-coordinate of the centroid is

$$\frac{V_z}{V}=\frac{\frac{37}{32}\pi}{\frac{3}{2}\pi}=\frac{37}{48}$$

2) Area is given by

$$A=\frac{1}{2}\oint -ydx+xdy=$$

$$=\frac{1}{2}\left[\oint_{(-3,5)}^{(-1,1)}(-ydx+xdy)+\oint_{(-1,1)}^{(1,3)}(-ydx+xdy)+\oint_{(1,3)}^{(3,2)}(-ydx+xdy)+\right.$$
$$\left.+\oint_{(3,2)}^{(5,3)}(-ydx+xdy)+\oint_{(5,3)}^{(-3,5)}(-ydx+xdy)\right]$$

Equation of line passing through points $(-3,5),(-1,1)$:

$$y=-2x-1$$

Then

$$A_1=\frac{1}{2}\oint_{(-3,5)}^{(-1,1)}(-ydx+xdy)=\frac{1}{2}\int_{-3}^{-1}(2x+1-2x)dx=\frac{1}{2}(-1+3)=1$$

Equation of line passing through points $(-1,1),(1,3)$:

$$y=x+2$$

Then

$$A_2=\frac{1}{2}\oint_{(-1,1)}^{(1,3)}(-ydx+xdy)=\frac{1}{2}\int_{-1}^{1}(x+2-x)dx=\frac{1}{2}2(1+1)=2$$

Equation of line passing through points $(1,3),(3,2)$:

$$y=-\frac{1}{2}x+\frac{7}{2}$$

Then

$$A_3=\frac{1}{2}\oint_{(1,3)}^{(3,2)}(-ydx+xdy)=\frac{1}{2}\int_{1}^{3}\left(\frac{1}{2}x-\frac{7}{2}-\frac{1}{2}x\right)dx=-\frac{7}{4}(3-1)=-\frac{7}{2}$$

Equation of line passing through points $(3,2),(5,3)$:

$$y=\frac{1}{2}x+\frac{1}{2}$$

Then

$$A_4 = \frac{1}{2}\oint_{(3,2)}^{(5,3)} (-ydx + xdy) = \frac{1}{2}\int_3^5 \left(-\frac{1}{2}x - \frac{1}{2} + \frac{1}{2}x\right) dx = -\frac{1}{4}(5-3) = -\frac{1}{2}$$

Equation of line passing through points $(5,3), (-3,5)$:

$$y = -\frac{1}{4}x + \frac{17}{4}$$

Then

$$A_5 = \frac{1}{2}\oint_{(5,3)}^{(-3,5)} (-ydx + xdy) = \frac{1}{2}\int_5^{-3} \left(\frac{1}{4}x - \frac{17}{4} + \frac{1}{4}x\right) dx = -\frac{17}{8}(-3-5) = 17$$

Therefore,

$$A = A_1 + A_2 + A_3 + A_4 + A_5 = 1 + 2 - \frac{7}{2} - \frac{1}{2} + 17 = 16$$

3) According to the Stokes' theorem,

$$\oint_{C_1 - C_2 - C_3} F \cdot ds = \iint_\Sigma |\mathbf{curl}_z(F)| d\Sigma,$$

where Σ is the shaded region. Since $|\mathbf{curl}_z(F)| = const = 9$, we obtain

$$\iint_\Sigma |\mathbf{curl}_z(F)| d\Sigma = |\mathbf{curl}_z(F)| \cdot A,$$

where A is the area of shaded region, which is

$$A = \pi(5)^2 - \pi(1)^2 - \pi(1)^2 = 23\pi$$

Therefore,

$$\oint_{C_1 - C_1 - C_1} F \cdot ds = \oint_{C_1} F \cdot ds - \oint_{C_2} F \cdot ds - \oint_{C_3} F \cdot ds = \iint_\Sigma |\mathbf{curl}_z(F)| d\Sigma$$

$$\oint_{C_1} F \cdot ds = \oint_{C_2} F \cdot ds + \oint_{C_3} F \cdot ds + |\mathbf{curl}_z(F)| \cdot A = 3\pi + 4\pi + 23\pi = 30\pi$$

Answer:

1)
   (A) $\left(-\frac{8}{11\pi} R \sin\frac{\pi}{12}, 0, \frac{H}{2}\right)$
   (B) $\left(\frac{\sqrt{3}}{\pi} h, 0, \frac{3}{4} h\right)$
   (C) $\left(\frac{1}{12}, 0, \frac{37}{48}\right)$
2) 16
3) 30π