Answer on Question #52133 - Math – Multivariable Calculus

1) Calculate the average value of the function $f(x,y) = e^{x+y}$ on the square $[0,1] \times [0,1]$

2) Calculate the average height above the x-axis of a point in the region $0 \le x \le x^2$

Solution

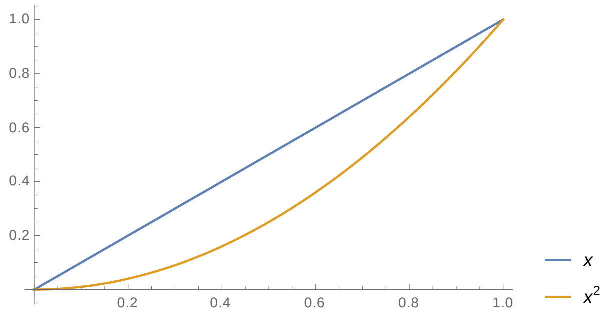
1) Let S be the area of the square $[0,1] \times [0,1]$. The average value of the function $f(x,y)$ on the square $[0,1] \times [0,1]$ is given by

$$\bar{f} = \frac{\iint_{0\,0}^{1\,1} f(x,y)dxdy}{S} = \frac{\iint_{0\,0}^{1\,1} e^{x+y}dxdy}{\iint_{0\,0}^{1\,1} dxdy} = \frac{\int_0^1 e^y \left(\int_0^1 e^x dx\right) dy}{\int_0^1 \left(\int_0^1 dy\right) dx} =$$

$$= \frac{\left(\int_0^1 e^x dx\right) \cdot \left(\int_0^1 e^y dy\right)}{\left(\int_0^1 dy \int_0^1 dx\right)} = \frac{(e^x|_0^1) \cdot (e^y|_0^1)}{(x|_0^1) \cdot (y|_0^1)} = \frac{(e^1 - e^0) \cdot (e^1 - e^0)}{(1-0) \cdot (1-0)} =$$

$$= (e-1)^2$$

2) The two curves $y = x$ and $y = x^2$ intersect at two points, where $x = 0$ and $x = 1$.



So the area of the region $D = \{(x,y): 0 \le x \le 1,\ x \le y \le x^2\}$ is given by

$$S(D) = \iint_D dxdy = \int_0^1 \left(\int_{x^2}^{x} dy\right) dx = \int_0^1 (y|_{x^2}^{x})dx = \int_0^1 (x - x^2)dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 =$$

$$= \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$$

The average height is given by

$$h = \frac{\iint_D ydxdy}{S(D)},$$

where the double integral is taken over the given region D.

The previous formula can be rewritten in the following way:

$$h=\frac{\iint_D \ ydydx}{S}=\frac{\int_0^1\left(\int_{x^2}^{x} ydy\right)dx}{S}=\frac{\int_0^1\left(\frac{y^2}{2}\Big|_{x^2}^{x}\right)dx}{S}=\frac{\frac{1}{2}\int_0^1(x^2-x^4)dx}{S}$$

$$=\frac{\frac{1}{2}\left(\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_0^1}{S}=\frac{\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}-0\right)}{\frac{1}{6}}=\frac{\left(\frac{1}{6}-\frac{1}{10}\right)}{\frac{1}{6}}=\frac{4\cdot 6}{60}=\frac{2}{5}=0.4$$

Answer

1) $(e-1)^2$
2) 0.4