Answer on Question #51992 – Math – Multivariable Calculus

1) Use the method of Language multipliers, find the maximum value of the function $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2xy + 2yz + zx = 12, x > 0, y > 0, z > 0$

Solution

Let

$$g\ (x, y, z) = 2xy + 2yz + zx - 12.$$

Compute all the points (x, y, z) that satisfy $\nabla f\ (x, y, z) = \lambda \nabla g\ (x, y, z)$ and the constraint $g\ (x, y, z) = 0$. The gradient vectors can be computed easily:

$$\nabla f\ (x, y, z) = \langle f_x, f_y, f_z \rangle = \langle yz, zx, xy \rangle.$$

$$\nabla g\ (x, y, z) = \langle g_x, g_y, g_z \rangle = \langle 2y + z, 2x + 2z, 2y + x \rangle.$$

Thus, we have the following four equations:

$$yz = \lambda(2y + z) \qquad (1)$$

$$zx = \lambda(2x + 2z) \qquad (2)$$

$$xy = \lambda(2y + x) \qquad (3)$$

$$2xy + 2yz + zx - 12 = 0 \qquad (4)$$

There are four unknowns and four equations. Hence we should be able to solve the system of equations. Multiplying both sides of (1) by *x* and both sides of (3) by *z* lead to

$$xyz = \lambda(2xy + xz)$$

$$xyz = \lambda(2zy + xz)$$

Subtracting these equations gives

$$0 = \lambda(2xy - 2zy) = 2\lambda y(x - z)$$

Hence, there are three possibilities, $\lambda = 0$ or $y = 0$ or $(z - x) = 0$.

By the statement of problem, $y > 0$, therefore $y = 0$ is impossible.

If $\lambda = 0$, then plugging to (1)–(3) gives $xy = yz = zx = 0$, thus equation (4) becomes

$2 \cdot 0 + 2 \cdot 0 + 0 - 12 = 0$, which leads to $-12 = 0$, which does not hold. It means that $\lambda \neq 0$.

Finally we must have $z = x$. Plugging $z = x$ into (2) leads to

$$x^2 = 4\lambda x,$$

hence $x = 4\lambda$, since $x \neq 0$. Plugging $x = 4\lambda$ into (3) gives

$$4\lambda y = \lambda(2y + 4\lambda) \rightarrow y = 2\lambda.$$

Therefore, we have $x = z = 4\lambda$ and $y = 2\lambda$. Finally, plugging these to the constraint equation (4) leads to

$$16\lambda^2 + 4 \cdot 2\lambda \cdot 4\lambda - 12 = 0 \to 4\lambda^2 = 1 \to \lambda = \pm\frac{1}{2}.$$

But we must have $x > 0, y > 0, z > 0$. Thus, $\lambda = \frac{1}{2}$.

This leads to

$$x = 2, z = 2, y = 1.$$

Since there is only one point, there is no need to compare the function value at this point with that of the other point. Therefore, the maximum volume is

$$f(2,1,2) = 2 \cdot 1 \cdot 2 = 4.$$

Answer: 4.

2) Use polar coordinates to find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

Solution

This volume is given by

$$\iint_D \sqrt{16 - x^2 - y^2} dA$$

where D is the domain

$D = \{(x, y) \mid 4 \le x^2 + y^2 \le 16\}.$

Besides, $-\sqrt{16 - x^2 - y^2} \le z \le \sqrt{16 - x^2 - y^2}$, apply the symmetry with respect to xOy plane.

Using polar coordinates $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$, we see that the volume is

$$2\int_0^{2\pi}\int_2^4 \sqrt{16 - r^2} r dr\, d\theta = 2\int_0^{2\pi} d\theta \int_2^4 r\sqrt{16 - r^2} dr = \theta|_0^{2\pi}\left(-\int_2^4 (-2r)\sqrt{16 - r^2} dr\right)$$

$$= (2\pi - 0)\int_2^4 -\sqrt{16 - r^2} d(16 - r^2)$$

$$= |t(r) = 16 - r^2, dt = -2rdr, t(2) = 12, t(4) = 0\,| = -2\pi\int_{12}^0 \sqrt{t} dt =$$

$$= 2\pi\int_0^{12} \sqrt{t} dt = 2\pi \frac{t^{3/2}}{3/2}\bigg|_0^{12} = \left(\frac{4\pi}{3} \cdot 12\sqrt{12} - 0\right) = 32\sqrt{3}\pi.$$

Answer: $32\sqrt{3}\pi$.

3) Given the vector field

$$F(x, y, z) = (y^2 - 4z, 2xy + 6yz, -4x + 3y^2)$$

a) Find a potential function of F

Solution

We first check if the vector field $F(x, y, z)$ is conservative by calculating its curl, which in terms of the components of $F(x, y, z)$ is

$$\mathbf{curl}\, F(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) =$$

$$= \left(\frac{\partial(-4x+3y^2)}{\partial y} - \frac{\partial(2xy\ +6yz)}{\partial z}, \frac{\partial(y^2-4z)}{\partial z} - \frac{\partial(-4x+3y^2)}{\partial x}, \frac{\partial(2xy\ +6yz)}{\partial x} - \frac{\partial(y^2-4z)}{\partial y}\right) =$$

$$= (6y - 6y, -4 - (-4), 2y - 2y) = (0,0,0)$$

The curl of $F(x, y, z)$ is zero. The vector field is defined in all $\mathbb{R}^3$, which is simply connected, so the vector field $F(x, y, z)$ is conservative.

Let $f(x, y, z)$ be a potential function of F. It means that

$$f_x = y^2 - 4z; f_y = 2xy\ + 6yz; f_z = -4x + 3y^2 \qquad (5)$$

where $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z}$.

Integrating the first equation in (5) with respect to *x*, we get

$$f(x, y, z) = xy^2 - 4xz + g(y, z). \qquad (6)$$

Differentiating (6) with respect to y, we obtain

$$f_y = 2xy + g_y(y, z) \qquad (7)$$

so comparing expressions for f_y in (5) and (7), we find that

$$g_y(y, z) = 6yz. \qquad (8)$$

Integrating (8) with respect to *y*, we get

$$g(y, z) = 3y^2z + h(z).$$

So, (6) becomes

$$f(x, y, z) = xy^2 - 4xz + 3y^2z + h(z) \qquad (9)$$

and differentiating (9) with respect to z gives

$$f_z = -4x + 3y^2 + h_z(z). \qquad (10)$$

so comparing expressions for f_z in (5) and (10), we find that

$$h_z(z) = 0$$

$$or$$

$$h(z) = const, \qquad (11)$$

where $const$ is an arbitrary real constant.

Taking into account (9) and (11), we find that

$$f(x, y, z) = xy^2 - 4xz + 3y^2z + const.$$

Answer: $\boldsymbol{xy^2 - 4xz + 3y^2z + const}$.